



Signature change in Spherical Vacuum spacetimes

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TO MY MOTHER AND LATE FATHER

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Conventions and Symbols

The Lorentzian signature used is $-+++$, and necessarily the signature is $++++$ in the Euclidean region.

Subscript $_e$ denotes evaluation in the Euclidean region.

Subscript $_l$ denotes evaluation in the Lorentzian region.

When unspecified, a general formula is valid in form in both spaces.

Greek indices are spatial (1,2,3), the Latin indices are of both space and time (0,1,2,3).

List of symbols:

V	-manifold
x^i	-4-space coordinates
g_{ik}	-4-metric of space
z^α	-junction surface coordinates
$s(x^i)$	-locus of junction surface
T_{ij}	-energy stress tensor
n_b	-normal to junction surface
$K_{\alpha\beta}$	-extrinsic curvature
G_{ij}	-Einstein tensor
\mathfrak{R}	-curvature invariant of 3-surface
R	-radial coord. in Schwarzschild metric
$R(r, t)$	-aerial radius in Tolman metric
T	-time coord. in Schwarzschild metric
M	-mass term in Schwarzschild metric
$M(r)$	-mass term in Tolman metric
e_1, e_2	-sign factors in Schwarzschild metric
e_3	-sign factor in normal to surface
n	-sign factor in Tolman metric
P^a	-4-momentum

Chapter 1

Introduction

The space-time in general relativity is usually considered to possess a Lorentzian signed metric. Positive definite Euclidean regions, with a Euclidean signed metric, have come into prominence lately through the Hartle and Hawking program concerning the wave function of the universe [10]; they utilize the fact that the Einstein Field Equations do not specify the signature of the metric, hence a Euclidean signed metric is possible. A general aim of that program is to try get a handle on the boundary conditions of the universe. An intriguing suggestion made in [10] is that the universe has no boundary, i.e. no origin where initial conditions have to be set. Another interesting development is the introduction of Euclidean wormholes. These wormholes can arise in one universe and connect it either to itself or to another universe. In order to attribute a transition probability, for example, between two Lorentzian regions, they require integrating the action along the tube connecting the two

regions under study. In normal Lorentzian space the path integral approach leads to oscillating behavior, and hence to non-convergence of the integral. To obtain convergence they resort to the transformation $t \rightarrow it$ which in effect introduces a Euclidean signature. This means in effect that we have two Lorentzian regions connected through a Euclidean region.

The Hartle and Hawking program and Quantum cosmology in general require that General Relativity will be formulated in a Quantum Mechanical fashion. They represent General Relativity in a Hamiltonian formulation, in which the dynamical degrees of freedom are the spatial components of the metric while the time is implicitly given through the particular choice in which the spatial three-surfaces cover the four-space. To obtain this 3+1 decomposition of the space-time they use the ADM decomposition method. The method involves using so called lapse and shift functions to relate adjacent three-surfaces. The lapse function being defined by the proper distance between two adjacent hypersurfaces, as measured along a normal to both surfaces, $d\tau = n(t)dt$. The shift function being the vector specifying the position of a point on one surface, with respect to the normal projection, of a point with the same surface coordinate from the adjacent surface. (See figure (1.1)).

The investigation of transitions between Schwarzschild geometries through a Euclidean region are also of interest when considered in conjunction with Smolin's idea [19]. Smolin's hypothesis is an attempt to motivate for the particular choice of fundamental constants governing physical interaction in our universe. In particular why are the fundamental constants tuned in such

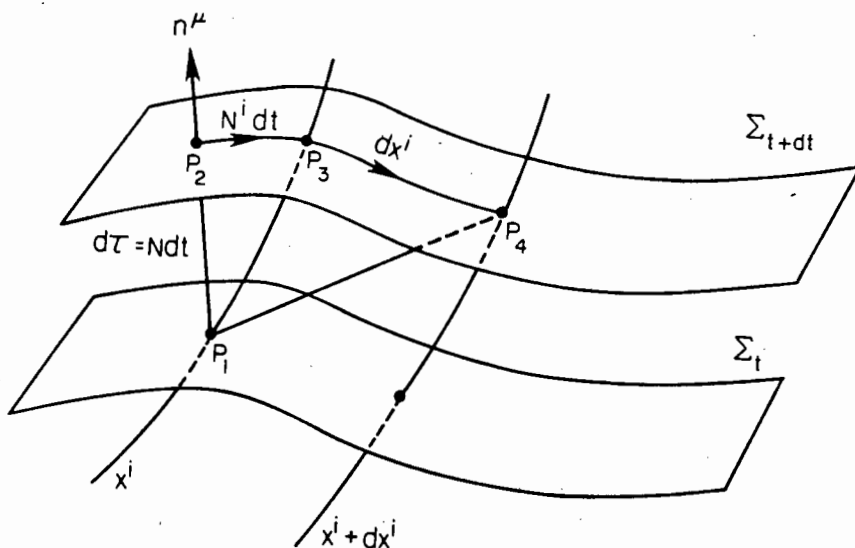


Figure 1.1: Lapse and shift functions.

a way in our present universe that they allow the existence of life. In his paper, life supporting characteristics are linked to the existence of stars whose abundance is linked to the abundance of black holes. Although there are some problems with this association [20], the general idea is interesting to pursue. In essence the idea is as follows. Each universe produces offsprings with small random differences in the values of their fundamental constants. The process of universe production occurs at each gravitational collapse singularity of the parent universe, be it a black hole or the final crunch. In his discussion Smolin limits his attention to compact universes, so each universe is assured of at least one offspring. After a sufficient time for many generations of universes, the universe type which will be the most abundant, will be the one whose fundamental constants ensure the most numerous production

of offsprings. This means that the most favoured type of universe is the one which gives rise to the largest number of black holes. The same criterion which favors universe reproduction, and ensures it being the most probable type of universe, happens to be also the one which is most likely to support life.

The Euclidean bounce in the Schwarzschild geometry, can provide a mechanism for the generation of the child universes. This suggests looking for a Euclidean bounce in two types of situations. The first is in the initial and final singularity of a universe and the second is a Euclidean bounce inside a black hole.

Paralleling the Quantum cosmology program, papers [1],[2] pointed out that the possibility of a change in the signature of the metric is not restricted to a quantum description of General Relativity. They showed that classical General Relativity, does not prevent the existence of Euclidean regions and then went on to produce some examples of signature change in the Robertson Walker metric. Even though the metric signature is invisible to the Einstein Field Equations, it should be noted that a change of signature is not, either g_{00} goes through zero, in which case the metric is degenerate there, or g_{00} jumps from a positive to a negative value, in which case the metric is discontinuous.

This thesis follows the approach of papers [1],[2] by exploring signature changes in other metrics. The metrics we chose to investigate are the Schwarzschild metric and the Tolman metric . The Schwarzschild metric was

originally chosen in order to investigate whether the neighborhood of the singularity inside a black hole can be replaced with a Euclidean region, and also to see whether this Euclidean region can lead to new universes by providing “wormholes” through to other Lorentzian universes. By this we mean that, if one follows “time-like” geodesic paths from a Lorentzian region into a Euclidean region, they bounce (instead of hitting a singularity) and can then pass through a second signature change into another Lorentzian region. Consideration of how geodesics pass through a signature change naturally leads to the Tolman metric, whose vacuum cases cover the Schwarzschild/Kruskal-Szekeres manifold with all possible sets of radial geodesic coordinates. We take the opportunity to explore several cases of signature change in other Tolman models.

The first step one has to take is to ensure that the various regions composing the space match geometrically. In chapter two we describe the three matching schemes commonly used. We mention the relation between them and then motivate our choice of the Darmois matching conditions. Having selected the Darmois matching conditions we describe the details of the matching. In particular we note the adjustments from the standard fixed signature matching which are needed to match regions with different signatures. In chapter three we describe how paper [1] approaches the task of matching two regions with different signatures in classical GR. We summarize only the sections relevant to the flow of this thesis. The entire paper is reproduced in the appendix.

Having put the mechanism for matching at a signature change in place,

chapter **four** deals with the case of signature change in the Schwarzschild metric. We introduce two signature factors into the original Schwarzschild metric, and obtain the corresponding Einstein Field Equations(EFE). From all the signature possibilities we select the ones of interest to us and then apply the geometrical matching conditions for two types of surfaces. We proceed to find the geodesics of each region and then investigate the paths of the geodesics in the combined space.

In chapter **five** we investigate signature changes in the Tolman metric. The vacuum Tolman model can, with the right parameters, describe the complete spherically symmetric vacuum space-time, and it does so with well behaved coordinates, which simplify analysis. After investigating the Schwarzschild-like option we investigate other vacuum and non-vacuum Tolman models which undergo signature change.

The matching details and case studies in chapters 4 and 5, are original work in the topic.

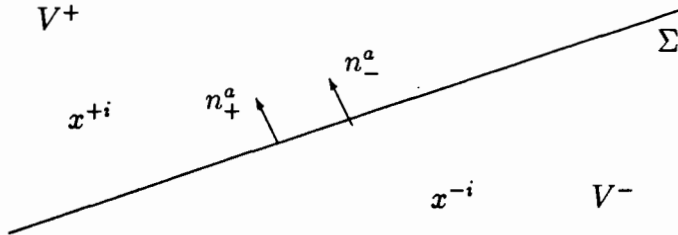
Chapter **six** contains concluding remarks evaluating the material covered and suggestions for future investigation.

Chapter 2

Matching conditions

In this chapter we give a brief description of the three commonly used types of matching conditions. This is followed by motivation for the selection of the Darmois matching conditions and how they are used.

Our space is divided into two sections with a boundary surface Σ joining the two. We denote the two regions separated by the boundary surface as V^+ and V^- . We give each region a coordinate chart. One chart we denote as x_+^i and the other as x_-^i with metrics g_{ab}^+ and g_{ab}^- respectively. Setting the intrinsic coordinates of the junction surface to be z^α , we then assume the locus of the surface is given parametrically in V^+ and V^- by $x_+^i = h_+^i(z^\alpha)$ and $x_-^i = h_-^i(z^\alpha)$ (Latin indices cover 0,1,2,3, and Greek indices 1,2,3.)



In each region the junction surface may also be described by $s^+(x^{+i}) = 0$ and $s^-(x^{-i}) = 0$ respectively. We also use the following notation: $|^\pm$ denotes evaluation of a quantity in the limit as the surface is approached from either region V^+ and V^- . $[\]$ denotes the difference between the values of the enclosed quantity as evaluated on either side of the surface, $|^+ - |^-$.

2.1 Matching conditions compared

The most widely used matching condition is that called the O'Brien and Synge condition, hereafter 'OS' [13]. They are derived by investigating the jump conditions of the metric and energy stress tensor across a surface layer in the limit where the surface layer thickness goes to zero. In this matching scheme we choose the coordinates such that the junction surface is given by $x^0 = \text{constant}$. The two regions separated by the junction surface match if all the metric components g_{ik} , the normal derivatives of the non-normal metric components $\frac{\partial g_{\mu\nu}}{\partial x^0}$ and the normal components of the energy stress tensor T_k^0 are continuous across the junction surface. (It has been pointed out [21] that the continuity of the energy stress tensor component is already assured if the

conditions on the metric are satisfied).

The second set of conditions are the Lichnerowicz conditions (hereafter ‘L’) [14]. This set of conditions is derived from purely mathematical considerations concerning the properties of manifolds, namely which coordinates can transmit correctly the differential structure of the manifold. These conditions require that at every point on the junction surface one be able to set up an admissible coordinate system in the neighborhood of that point, so that the metric and the metric’s first derivatives are continuous. An example of such an admissible coordinate system is the Gaussian coordinates.

The Darmois conditions (hereafter ‘D’) [8] require the continuity of the first and second fundamental forms of the junction surface.

2.2 Darmois matching

We now describe in more detail how to use the Darmois matching conditions in the normal case of a constant signature boundary. The information we require is the metrics on either side of the junction and a parametric description of the junction surface. The first fundamental form is the intrinsic metric of the junction surface. The second fundamental form is the extrinsic curvature of the surface. The extrinsic curvature of a surface is the projection onto the surface of the rate of change in the direction of the normal to that surface in the enveloping space. These forms are evaluated on either side of the surface using the metric of either region and are then compared.

Surface Metric

The intrinsic metric is obtained by projecting the 4-metric onto the surface by using the basis vectors of that surface $e_\alpha^a = \frac{\partial x^a}{\partial z^\alpha}$.

$$g_{\alpha\beta}^{(3)} = g_{ab}^{(4)} \frac{\partial x^a}{\partial z^\alpha} \frac{\partial x^b}{\partial z^\beta}$$

Extrinsic Curvature

Extrinsic curvature is defined for surfaces which have an embedding space. Consider the normal to the surface in the enveloping space. Extrinsic curvature is the linear operator which gives the rate of change of direction of this normal on the junction surface (with respect to the surface coordinates). So the extrinsic curvature describes the surface's shape in the enveloping space. This means that if we want to match two surfaces, their shape and hence extrinsic curvature must match.

To determine the extrinsic curvature ($K_{\alpha\beta}$) of the surface given by $s(x^a) = \text{constant}$, we use the normal to this surface, given by

$$n_b = \frac{\sqrt{\epsilon_3} \partial s(x^a) / \partial x^b}{\sqrt{\partial^a s \partial_a s}} \quad (2.1)$$

where $\epsilon_3 = n^b n_b = +1$ if Σ is time-like and -1 if it is space-like. An important point to note, is that the normals (i.e. n_\pm^b) have to point from V^- to V^+ on either side of the surface, this ensures proper comparison of the extrinsic curvature of the two surfaces.

To obtain the extrinsic curvature we now project the gradient of the

normals onto the junction surface:

$$K_{\alpha\beta} = (\nabla_i n_j) \frac{\partial x^i}{\partial z^\alpha} \frac{\partial x^j}{\partial z^\beta} \quad (2.2)$$

This expression is not always convenient to use, so we expand:

$$K_{\alpha\beta} = \left(\frac{\partial n_j}{\partial x^i} - \Gamma_{ij}^k n_k \right) \frac{\partial x^i}{\partial z^\alpha} \frac{\partial x^j}{\partial z^\beta} \quad (2.3)$$

and use the fact that the projection of n_i onto the surface is zero by definition

$$\frac{\partial x^j}{\partial z^\beta} n_j = 0 \rightarrow \frac{\partial}{\partial x^k} \left(\frac{\partial x^j}{\partial z^\beta} n_j \right) = 0 \quad (2.4)$$

hence

$$\frac{\partial n_j}{\partial z^\alpha} \frac{\partial x^j}{\partial z^\beta} = -n_j \frac{\partial^2 x^k}{\partial z^\alpha \partial z^\beta} \quad (2.5)$$

Which leads to the following easily workable expression for the extrinsic curvature:

$$K_{\alpha\beta} = -n_k \left(\frac{\partial^2 x^k}{\partial z^\alpha \partial z^\beta} + \Gamma_{ij}^k \frac{\partial x^i}{\partial z^\alpha} \frac{\partial x^j}{\partial z^\beta} \right) \quad (2.6)$$

The Darmois conditions are now given as:

$$[K_{\alpha\beta}] = 0 \quad \& \quad \left[g_{\alpha\beta}^{(3)} \right] = 0 \quad (2.7)$$

In the constant signature case, according to [15], the Darmois conditions are equivalent to the Lichnerowicz matching conditions, whereas the OS conditions are more restrictive than Darmois and thus admit a smaller class of allowable transitions.

The matching conditions described above are the conditions used when there is no signature change across the boundary. The question which arises

now is which matching conditions can be used in the presence of a signature change, and whether they require any modification.

The equivalence between D and L breaks down when we introduce signature change. Both the Lichnerowicz and OS conditions insist that all the metric components be matched on either side of the junction surface. This insistence, which will be discussed further in chapter three, inevitably leads to a degenerate metric and a non-affine time coordinate.

We select the Darmois matching conditions as they are the only conditions which do not require the continuity of the normal component of the metric, g_{00} , thus permitting a change of signature if x^0 is the time direction with the parameter proper time (distance). Also they are invariant to the coordinates chosen on either side. In fact they are blind to the change of signature, providing the surface is space-like in the Lorentzian space-time, requiring no modification whatever, thus extending the signature blindness of the EFEs.

The transition surface is space-like in both regions, but in the Euclidean region all vectors are necessarily space-like. This implies that now $n_a n^a = \epsilon_3$ has the value $\epsilon_3 = +1$ in the Euclidean region, and $\epsilon_3 = -1$ in the Lorentzian region. A point to note is that ϵ_3 is not a new independent quantity but rather a notational convenience. In effect it is equal to the negative of the sign factor to be used for changing the signature of the various metrics. In the Schwarzschild case it is the same as $-\epsilon_1$ in the constant time transition, $-\epsilon_2$ in the constant radius transition and $-n$ in the Tolman case.

In order to understand better the nature of the boundary surface it is

instructive to review Israel's study of the physical implications of junction conditions [12]. Israel presents an approach based on the D matching conditions, thus the surface of transition is described in a manifestly invariant geometrical fashion by using the extrinsic curvature of the surface and its imbedding in space-time.

The momentum and energy density are expressed as a function of the extrinsic curvature.

$$-2\epsilon_3 G_{ij} n^i n^j = {}^3R - \epsilon_3 (K_{\alpha\beta} K^{\alpha\beta} - K^2) \quad (2.8)$$

$$G_{ij} e^i_\alpha n^j = K^\beta_{\alpha|\beta} - K_{|\alpha} \quad (2.9)$$

\mathcal{R} is the intrinsic curvature invariant of the 3-surface, $K = g^{\alpha\beta} K_{\alpha\beta} = K^\alpha_\alpha$ and $_{|}$ denotes covariant derivatives in the 3-surface.

If the Darmois matching is imposed, the requirement on the extrinsic curvature $K^-_{\alpha\beta} = K^+_{\alpha\beta}$ leads to both the energy density and the momentum density (as measured by an observer moving orthogonally to Σ) being conserved across a space-like junction surface.

$$[G_{ij} n^i n^j] = 0 \quad (2.10)$$

$$[G_{ij} e^i_\alpha n^j] = 0 \quad (2.11)$$

In this form, Israel's identities are a generalized form of OS matching condition. The above expressions (2.8) and (2.9), are valid for constant signature transition. Reference [4] investigates the validity of conservation laws at a change of signature. The point of departure of [4] is to claim that the standard divergence theorem used for establishing continuity, is not valid across

a metric signature change. In particular the divergence theorem applied to the Einstein tensor

$$\oint_S G^{ij} v_j m_i d^3 S = \int_W \nabla_i (G^{ij} v_j) d^4 W \quad (2.12)$$

where v_j is a smooth vector field, and m_i is the unit normal to the boundary S of W , continues to hold across Σ if the signature does not change, thanks to Israel's result. The divergence theorem holds separately on each side of Σ , and the Israel identities are exactly what is needed to preserve conservation through Σ .

They then take account of the change in signature in the derivation of Israel's conditions. In his analysis ϵ_3 remains fixed across the junction surface, but in our case ϵ_3 changes from -1 (Lorentzian) to $+1$ (Euclidean). They thus obtain from (2.8,2.9) the modified Israel identities

$$[G^{ij} n_i n_j] = -^3 R \quad (2.13)$$

$$\left[G_j^i n_i \frac{\partial x^j}{\partial \xi^\alpha} \right] = 0 \quad (2.14)$$

Hence they demonstrate that conservation fails, in the sense that the left hand side and right hand side of (2.12) are not any more equal, but show that this can be corrected by the insertion of a surface term defined on the transition surface, Σ , to the rhs of (2.12). Equations (2.13) and (2.14) show that momentum density is still conserved across the boundary as before, but energy density appears to have a new conservation law. It is suggested not to treat the jump in the energy density across the boundary as a result of a surface layer, but rather consider it an effect of the change in the physical nature of space.

It is apparent that no special modification of the Darmois matching conditions are necessary to adapt them for signature change. The only restriction immediately apparent for the matching across a signature change, is that the junction surface must necessarily be space like.

The above discussion motivates for a small difference in approach to matter conservation, between this thesis and that of Ellis et al paper [1](discussed in the next chapter). This thesis follows the spirit of [4] where it is claimed that, since the standard notion of energy conservation breaks down at a signature change, the imposition of further restrictions on the matching based on energy conservation criteria may not always be physically justified. The D conditions are taken as the essential minimum conditions for matching, giving the most general results while any extra conditions specialize to particular physical scenarios. In [1] the normal energy conservation criterion is applied to the surface of transition. Compliance with that criterion necessarily requires that any admissible solution would be free of distributional parts to the Fridman equation on that surface. It is pointed out in [4] that, provided D are satisfied, Israel's definition for the surface stress is zero, even if energy conservation does not hold in the normal sense. However a discontinuity in the energy- stress-tensor is unavoidable unless Σ is Ricci flat.

Chapter 3

Robertson Walker case review

In [1] the Robertson Walker metric is chosen for the purpose of investigation into signature changes, as this model describes the large scale evolution of the classical universe and thus parallels the Quantum cosmology attempt to describe the large scale evolution of the universe by means of a wave function of the universe. The line element given is

$$ds^2 = -n(t) dt^2 + R(t) \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right)$$

where the only factor different from the normal Robertson Walker line element is the lapse function $n(t)$. The lapse function introduces the signature change by changing sign itself. Two different approaches to matching the spaces are attempted in that paper. The first involves using a lapse function that changes sign smoothly by passing through zero at the junction surface. The second method is to have a discrete jump in the value of the lapse function across the boundary. The first approach insists on the continuity of the

full 4-metric through the signature change à la Lichnerowicz. This is done in order to maintain some sense of continuity of the metric despite its being degenerate on the surface. A discussion of the nature of the singularity in the metric exposes the fact that the differential structure forced by that choice of lapse function is not unique, different families of differential structure occur from different types of limiting behavior of $n(t)$ on the surface of transition. A unique differential structure across the junction is achievable if we use the continuity of a physical parameter as our matching criterion. This suggests the use of the proper time ($\tau = \int \sqrt{-ds^2}$ in the Lorentzian case) and proper distance ($d = \int \sqrt{ds^2}$ in the Euclidean case) variables for that physical parameter. Another observation is that if we have a well defined differential structure, as suggested by using the proper time/distance approach, it is not equivalent to the differential structure obtained through any continuous lapse function choice, since the latter is degenerate at the surface and hence can not transform to the first. The physical matching criterion rather than metric continuity is selected as the more realistic. They then proceed to apply what are effectively the Darmois junction conditions, to ensure the geometrical matching of the two spaces. It is then observed that the evolution of the solution as described by proper time/distance parameter is smooth through the transition. In this scheme the lapse function has a discontinuity across the boundary (also the metric) and thus can approach arbitrary values on either side of the junction. Rather than performing the matching using an arbitrary lapse function in either region they chose to fix the lapse at a constant value on either region $n = \pm 1$, thus transferring the matching to that of matching the coordinates only, eliminating the need to match another

arbitrary function.

Details are given in the appendix. We see there that a wide variety of behaviors is possible, including classical analogues of the Hartle- Hawking ‘no boundary’ beginning of the universe.

Chapter 4

The Schwarzschild case

In this chapter we investigate signature change in the Schwarzschild metric. To clarify the properties of the Euclidean sections with which we will deal shortly, it is of benefit to review initially some relevant properties of the standard Schwarzschild space. The usual line element is given as:

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dT^2 + \left(1 - \frac{2M}{R}\right)^{-1} dR^2 + R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2$$

It is convenient to divide the space into two sections, the outer solution $R > 2M$ describing the external solution of a spherically symmetric static star, and the inner solution $R < 2M$ used to describe the extreme gravitational properties of a black hole. At $R = 2M$ the metric is singular and describes a null 3-surface which separates the two regions. The sign of the g_{TT} and the g_{RR} components interchange across $R = 2M$, thus leading to reinterpretation of the roles of R and T . In both the outer and inner solutions R represents the areal radius, but in the outer solution it is a spatial coordinate in a static

space-time, while in the inner solution it is a time coordinate so the metric no longer represents a static solution.

To achieve greater clarity regarding the nature of the Schwarzschild geometry we transform into Kruskal-Szekeres coordinates.

The line element for Kruskal-Szekeres metric is given by:

$$ds^2 = (32M^3/R) e^{-R/2M} (-dv^2 + du^2) + R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (4.1)$$

where the Schwarzschild R is now a function of u and v defined implicitly by

$$(R/2M - 1) e^{R/2M} = u^2 - v^2 \quad (4.2)$$

In these coordinates a manifold unfolds which is larger than the Schwarzschild manifold, and in effect describes two outer and two inner Schwarzschild regions. Figure 4.1 shows the relation between the Schwarzschild and Kruskal-Szekeres coordinates.

Observing $T = \text{const.}$ spacelike sections through the K-S space-time reveals that we have two asymptotically flat spaces connected through a throat i.e a minimum of the areal radius R on that slice, there being no spherical origin $R = 0$. A qualitative observation of the dynamical behavior of space-like hypersurfaces, reveals that the the two universes start disconnected, each possessing a singularity at $R = 0$. A throat is formed at some stage, and it evolves to a maximum radius of $2M$, before pinching off again to leave the two universes disconnected again. (When the throat has a radius of $2M$ only an external solution exists).

We now turn our attention to imposing the signature change. Since there

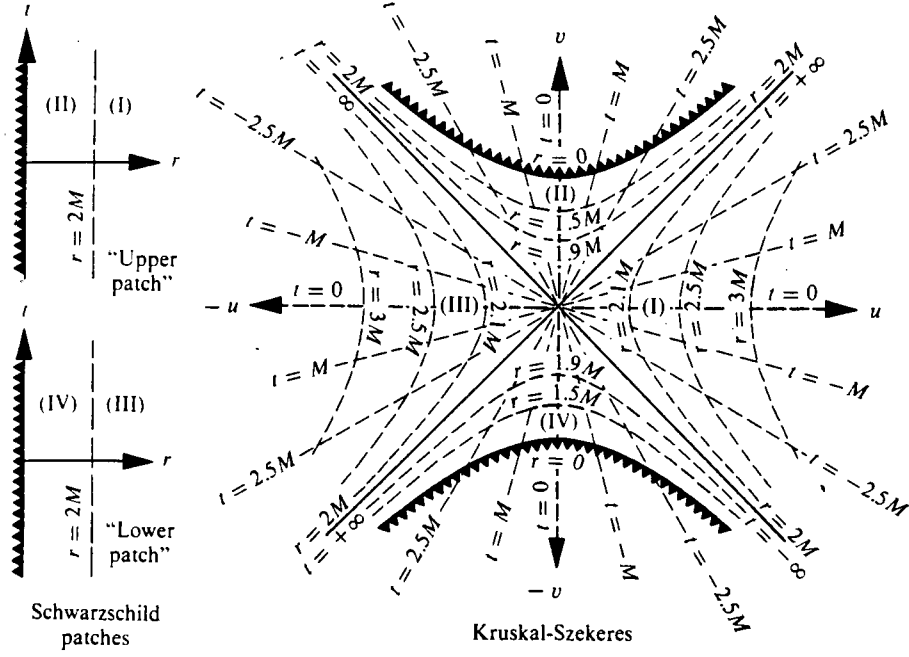


Figure 4.1: Kruskal-Szekeres coordinate system.

are two metric components which change sign at $R = 2M$, we introduce a new sign factor in each of the two. The first in the g_{TT} term and the second in the g_{RR} term. The new metric reads:

$$ds^2 = -\epsilon_1 \left(1 - \frac{2M}{R}\right) dT^2 + \epsilon_2 \left(1 - \frac{2M}{R}\right)^{-1} dR^2 + R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2 \quad (4.3)$$

Where $\epsilon_1 = \pm 1$ & $\epsilon_2 = \pm 1$ are the sign factors added to the original Schwarzschild metric to effect a signature change.

We now have to sort through all the new possibilities these factors introduce into the signature of the metric. It can be observed from the metric (4.3) that the sign of the g_{TT} and g_{RR} elements reverse across $R = 2M$. This introduces two different Lorentzian metrics - the standard one with $\epsilon_1 = 1$, $\epsilon_2 = 1$ and a new one (non-vacuum) with $\epsilon_1 = -1$, $\epsilon_2 = -1$ (referred to as “mod-

$r > 2M$		
$\begin{array}{c} \epsilon_2 \\ \hline \epsilon_1 \end{array}$	+1	-1
+1	Schwarzschild	Double Lorentzian
-1	Euclidean	Modified Lorentzian

$r < 2M$		
$\begin{array}{c} \epsilon_2 \\ \hline \epsilon_1 \end{array}$	+1	-1
+1	Schwarzschild	Euclidean
-1	Double Lorentzian	Modified Lorentzian

Figure 4.2: Possible metrics for different choices of ϵ_1 and ϵ_2 . Arrows indicate the desired transitions.

ified Lorentzian” in figure 4.2). We are interested only in transitions from the standard Schwarzschild metric to a Euclidean region, so we disregard the signature combinations which give us “double Lorentzian” (two time-like components) regions or the non-Schwarzschild Lorentzian metric.

Now we first obtain the Einstein Field equations from the new metric. Then we attempt matching across the two types of surfaces : a constant time surface and a constant radius surface. For each case we obtain the geometrical matching conditions and then look at the behavior of the geodesics to first ensure that each region is geodesically complete and secondly that all geodesics can be matched on the surface of transition.

4.1 Einstein Field Equations

Now that we have introduced these new sign factors into the metric parameters we need to re-derive the EFE to take account of this change. Solving the EFE for this metric leads to the following Einstein tensor components

$$G_{00} = \frac{\epsilon_1 (\epsilon_2 - 1) (R - 2M)}{\epsilon_2 R^3} \quad (4.4)$$

$$G_{11} = \frac{\epsilon_2 - 1}{R (2M - R)} \quad (4.5)$$

$$G_{22} = 0 \quad (4.6)$$

$$G_{33} = 0 \quad (4.7)$$

A vacuum solution requires ϵ_2 to be +1, and vacuum to vacuum transitions are therefore caused by a change in the sign of ϵ_1 . Transitions requiring a change of

sign in the ϵ_2 factor introduce non-vacuum solutions with strangely behaved matter (anisotropic pressure with only a radial component). From the Riemann tensor R^{abcd} we can calculate the Kretschmann scalar for this metric $k = R^{abcd}R_{abcd}$:

$$k = \frac{24 [(\epsilon_2 - 1) R (2M - R) + 6M^2]}{3R^6} \quad (4.8)$$

It appears that the Kretschmann scalar is only affected by the sign factor ϵ_2 . The Kretschmann scalar is singular only at $R = 0$, which implies that at $R = 2M$ there is only a coordinate singularity and not a physical one. Also, the singularity at $R = 0$ occurs irrespective of the sign of ϵ_2 , thus any Euclidean section containing $R = 0$ will have a singularity. We might be able to avoid it if we can establish that the $R = 0$ does not occur in the Euclidean space.

4.2 Imposing matching conditions

We now try to find surfaces on which signature change matching is possible. The first requirement for the transition surface, as mentioned in chapter two, is that it be a space-like surface. We find two such surfaces immediately by inspecting figure 4.2 and the metric (4.3). In the $R > 2M$ it is a constant T surface, and in $R < 2M$ it is a constant R surface. Selecting these surfaces also simplifies the calculations because the surface coordinates z^α may then be chosen to be identically 3 of the x^a coordinates in V^- and V^+ . Since we are matching two regions with different metrics, we use the matching conditions to establish the relation between the coordinates on either side of the junction.

4.2.1 Constant T surface

The intrinsic metric for the constant time surface, is given in both regions by

$$d\sigma^2 = \frac{\epsilon_2 R}{R - 2M} dR^2 + R^2 d\Omega^2 \quad (4.9)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

Our requirement for a standard Schwarzschild solution in the Lorentzian region sets the sign of ϵ_2 to +1, and this in turn requires that $R \geq 2M$ for a space-like surface. Although (4.9) is singular at $R = 2M$, all $T = \text{constant}$ surfaces intersect that point, which is only a coordinate singularity. In this transition ϵ_1 changes across the surface. Requiring the angular coordinates on either side to coincide ($\theta_e = \theta_l$ & $\phi_e = \phi_l$), also fixes the radius to be the same on either side ($R_e = R_l$). From the (g_{RR}) term of the surface metric it is also evident that the mass terms also coincide ($M_e = M_l$), hence the mass is the same in both regions.

This solution corresponds to vacuum in both regions. The extrinsic curvature of this surface is zero ($K_{ij} = 0$), so no further constraints are encountered. Since all $T = \text{constant}$ surfaces are equivalent (for a static metric), this result is not surprising.

4.2.2 Constant R surface

The simplest non-vacuum case occurs for a transition across a constant R surface.

The intrinsic metric of this surface is:

$$ds^2 = -\epsilon_1 (1 - 2M/R) dT^2 + R^2 d\Omega^2$$

If we require that the Lorentzian region be the usual Schwarzschild metric, then $\epsilon_1 = +1$ which limits where such a transition is possible to $R < 2M$, and in this

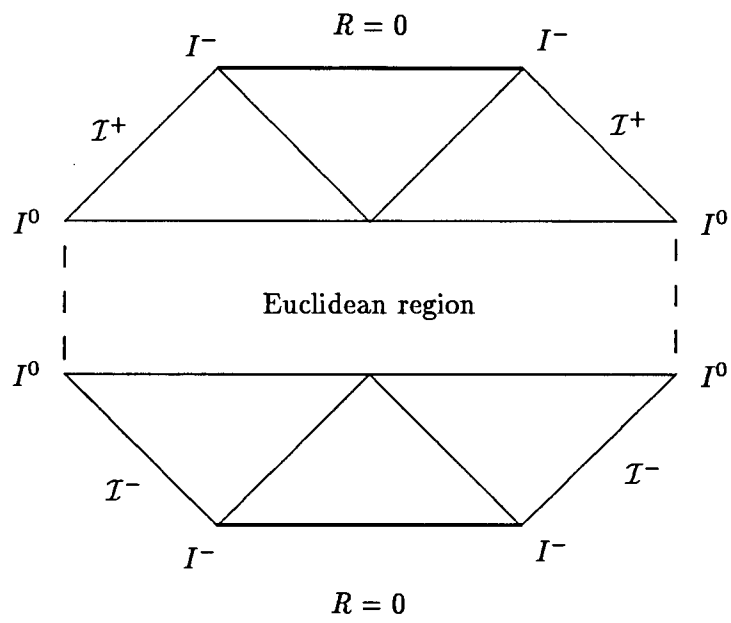


Figure 4.3: Penrose diagram for constant T transition.

case ϵ_2 changes across the surface. The extrinsic curvature of this surface ($\epsilon_1 = 1$) is given on both sides as:

$$K_{TT} = \frac{-M\sqrt{\epsilon_3(R-2M)}}{R^{\frac{5}{2}}\sqrt{\epsilon_2}}, \quad K_{\theta\theta} = \frac{\sqrt{\epsilon_3 R(R-2M)}}{\sqrt{\epsilon_2}}, \quad K_{\phi\phi} = \sin^2 \theta K_{\theta\theta}$$

From the discussion following the definition of the normal (2.1), we recall that the sign factor ϵ_3 is determined by the signature of the space. For $\epsilon_2 = +1$ (Lorentzian metric) $\epsilon_3 = -1$, and for $\epsilon_2 = -1$ (Euclidean metric) we have $\epsilon_3 = +1$. From this it is apparent that the ratio $\epsilon_3/\epsilon_2 = -1$ is the same in both the Euclidean and the Lorentzian regions. This leads to the expression for the extrinsic curvature being identical on either side:

$$K_{TT} = \frac{-M\sqrt{(2M-R)}}{R^{\frac{5}{2}}}, \quad K_{\theta\theta} = \sqrt{R(2M-R)}, \quad K_{\phi\phi} = \sin^2 \theta K_{\theta\theta}$$

From the intrinsic metric it is apparent that if we match the angular part ($\theta_e = \theta_l$ & $\phi_e = \phi_l$) then R must also match ($R_e = R_l$). Also from the matching of $g_{\theta\theta}$ we see that the mass terms ($M_e = M_l$) must match. The matching conditions imposed by the intrinsic metric also ensure the extrinsic curvature matching and no further restrictions are necessary.

This demonstrates that matching can be achieved on a surface that is entirely inside the horizon, but since the Euclidean region is not empty in this case, it still leaves open the interpretation of the energy stress tensor in the Euclidean side.

4.3 Geodesics

So far we have applied the Darmois junction conditions for two different surfaces. The geometrical matching assured us that the two regions can be matched, but we still lack information on what particle paths look like in the combined space. To

obtain those particle orbits we now investigate the behavior of geodesic paths inside those spaces. One aim is to verify that the each space is geodesically complete. Once we establish the completeness of each region we move on to identify each geodesic which originates in one region and arrives at the transition surface to a geodesic in the other region. Once the identification is established for each geodesic which arrives at any point on the surface, the resulting set of composite geodesic paths describes particle paths in the combined space. Matching the geodesics is achieved by matching, first the R coordinate of the geodesic at the surface, this is already established from the geometrical matching conditions. The second, non-trivial criterion of matching is that we need to establish a sense of continuity for the information conveyed in the geodesic tangent vector across the boundary. This task is complicated by the fact that we are matching time-like geodesics on the Lorentzian side to space-like geodesics in the Euclidean side. This means that we would not be able to match both the T and R components of the tangent vectors (u^a) of the geodesics as well as their magnitudes, as is generally the case in standard matching without signature change.

The transition surface has a well defined metric, hence a well defined connection structure. This means that we can expect to be able to carry a vector tangent to this surface, from V^- to V^+ unambiguously. This motivates that as a first attempt at matching geodesic tangent vectors, we match the components which are tangential to the junction surface and the absolute values of the magnitudes of the vectors, which are already unit vectors.

In the second attempt we match the components of the tangent vector which are normal to the surface and their magnitudes.

The third attempt involves matching both components of the 4- momentum

(P^a) and then inferring the particle mass change across the junction surface. This approach is motivated if we require that the momentum flux be continuous through the junction. Israel's identity (2.14), which states that the momentum flux is conserved across the transition surface, lends support to this approach.

We use the Euler-Lagrange equation and the magnitude condition of the Lagrangian to obtain the tangents to the geodesics. From the Lagrangian of this metric:

$$\begin{aligned}\mathcal{L} &= g_{ab} \frac{dx^a}{d\tau} \frac{dx^b}{d\tau} \\ &= g_{ab} u^a u^b \\ &= 1/2 \left\{ -\epsilon_1 (1 - 2M/R) \dot{T}^2 + \epsilon_2 (1 - 2M/R)^{-1} \dot{R}^2 + R^2 \dot{\theta}^2 + R^2 \sin^2 \theta \dot{\phi}^2 \right\}\end{aligned}\tag{4.10}$$

where $\dot{} \equiv \frac{d}{d\tau}$.

The Lagrangian terms involving θ and ϕ are unaffected by the signature change so we concentrate on solving for radial geodesics only, we thus set $\dot{\theta} = \dot{\phi} = 0$. We now derive the T component of the Euler-Lagrange Equations for radial geodesics:

$$\frac{d}{d\tau} \left[-\epsilon_1 (1 - 2M/R) \dot{T} \right] = 0\tag{4.11}$$

The magnitude condition is $2\mathcal{L} = 1$ in a Lorentzian region and $2\mathcal{L} = -1$ in a Euclidean region.

We now solve the Euler-Lagrange Equations to obtain the radial geodesic tangent vectors in each of our regions.

From (4.11) we get the T component of the geodesic tangent vector

$$-\epsilon_1 (1 - 2M/R) \dot{T} = h\tag{4.12}$$

where h is a constant parameter for each geodesic.

Radial Geodesics (Lorentzian case)

The magnitude condition (4.10) becomes:

$$-1 = -(1 - 2M/R)\dot{t}^2 + (1 - 2M/R)^{-1}\dot{R}^2 \quad (4.13)$$

Using this equation with (4.12) one arrives at the tangent vector to the geodesic $u^a \equiv dx^a/d\tau$.

$$u^a = \left(\frac{-h_l}{1 - 2M/R}, \pm \sqrt{2M/R - 1 + h_l^2}, 0, 0 \right)$$

Inspecting this tangent vector we can distinguish three types of geodesics:

- (i) $1 - 2M/R \leq h_l^2 < 1$: Geodesics which have an extremum at $R = 2M/(1 - h_l^2)$. The radial coordinate acceleration is always negative, $\ddot{R} = -M/R^2$, and thus the extremum is a maximum.
- (ii) $h_l^2 = 1$: Geodesics which arrive at or fall from $R = \infty$ with zero velocity. These are monotonically outgoing or monotonically ingoing.
- (iii) $h_l^2 > 1$: Geodesics with finite velocity at $R = \infty$. These are also monotonically ingoing or outgoing.

Radial Geodesics (Euclidean case, $R \geq 2M$)

In this case the magnitude condition (4.10) reduces to :

$$1 = (1 - 2M/R)\dot{T}^2 + (1 - 2M/R)^{-1}\dot{R}^2$$

and the tangent to the geodesic becomes

$$u^a = \left(\frac{h_e}{1 - 2M/R}, \pm \sqrt{1 - 2M/R - h_e^2}, 0, 0 \right)$$

In this case there is only one type of geodesic. The acceleration of all geodesics is positive for all R , $\ddot{R} = M/R^2$, thus indicating that all have local extrema and those extrema are minima in R . The minima occur at $u^R = 0$ which means that $R \geq \frac{2M}{1-h_e^2}$. Reciprocally this means that the allowed range of h_e is

$$0 \leq h_e^2 \leq 1 - 2M/R \quad (4.14)$$

As can be observed, all geodesic paths are restricted to the region $R \geq 2M$. The only geodesic reaching $R = 2M$ is the one with $h_e = 0$ which in effect is a stationary point. This is in accord with our restriction on the space section, and confirms that the region $R \geq 2M$ is a geodesically complete manifold.

Radial Geodesics, $R < 2M$

An analysis similar to the above leads to the following tangent vectors

In the Lorentzian case (same as for $R \geq 2M$):

$$u^a = \left(\frac{h_l}{2M/R - 1}, \pm \sqrt{2M/R - 1 + h_l^2}, 0, 0 \right)$$

In the Euclidean case (different from $R \geq 2M$):

$$u^a = \left(\frac{-h_e}{2M/R - 1}, \pm \sqrt{2M/R - 1 - h_e^2}, 0, 0 \right)$$

In both cases the acceleration is given by

$$\ddot{R} = -\frac{M}{R^2} \quad (4.15)$$

This means that even though a spatially contained matching surface was found, the geodesics in the matched Euclidean region do not bounce, but hit the curvature singularity at $R = 0$, see equation (4.8). Observing the Euclidean tangent vector

we see that R is restricted to the region $R \leq 2M$. This fact viewed together with the existence of maxima for all orbits, indicates that the Euclidean region $R \leq 2M$ is geodesically complete, and can be treated as a complete manifold.

4.3.1 Matching the tangential component

The tangent vector to the radial geodesic is described by three quantities, $\partial R/\partial\tau = u^R$, $\partial T/\partial\tau = u^T$ and the magnitude $u_a u^a$, where any two imply the third, given the signature. In our type of signature transition, only the metric elements tangential to the surface of transition are continuous. This suggests a matching of the tangential components of the geodesic tangent vectors at the transition surface. Naturally the magnitude jumps from -1 to +1. The tangent element orthogonal to the transition surface is then fixed in the second region, and in general it is not continuous across the junction surface.

constant T surface

We now match the $\partial R/\partial\tau$ component. The matching requires that the following condition be satisfied

$$h_l^2 + h_e^2 = 2 \left(1 - \frac{2M}{R} \right)$$

It can be immediately observed that the matching cannot be satisfied for all geodesics. At any given R value (for $R \geq 2M$), h_l can take a range of values for which matching is not possible for any h_e satisfying (4.14). This leads us to conclude that this matching criterion is unsatisfactory as it cannot continue all the Lorentzian geodesics that arrive at a signature change surface.

constant R surface

The tangential component is now the time component. Matching the time components of the tangent vectors leads to the requirement that $h_l = -h_e$, so again the range of h_l can not be completely covered by the permissible range of h_e in (4.14) hence the matching criterion is unsatisfactory.

4.3.2 Matching the normal component

Failure of the above conditions leads to attempt matching the normal components and the magnitude.

Constant T surface

Matching the u^T components leads to the requirement: $h_e = -h_l$. As pointed above this is unsatisfactory.

Constant R surface

Matching the u^R components leads to the requirement:

$$h_e^2 = -h_l^2 \rightarrow h_e = h_l = 0$$

. This is an even stronger restriction than the ones above and clearly is not satisfactory as a matching condition.

4.3.3 Matching the 4-momentum

In this scheme we match both the T and R components of the momentum vector. The 4-momentum, representing the momentum of a particle of rest mass m , is given in each region as:

$$P^a = mu^a \quad (4.16)$$

with necessarily

$$P^a P_a = +m_e^2$$

in the Euclidean region, and

$$P^a P_a = -m_l^2$$

in the Lorentzian region. We now match both momentum components across the surface

$$P_e^T = P_l^T \quad \& \quad P_e^R = P_l^R \quad (4.17)$$

Constant T surface

The matching conditions (4.17) lead to the following relations between the geodesic parameters h and m of either region

$$-h_l m_l = h_e m_e \quad (4.18)$$

$$m_l \sqrt{2M/R - 1 + h_l^2} = m_e \sqrt{1 - 2M/R - h_e^2} \quad (4.19)$$

We have only two equations relating the four quantities h_l, h_e, m_l and m_e . This freedom allows us to match any particle moving along the geodesic with, say in the Lorentzian region with h_l^Σ and m_l^Σ to a geodesic in the Euclidean region, by choosing the required h_e^Σ and m_e^Σ to satisfy (4.18) and (4.19).

The nature of h_l is well understood in the Lorentzian region so we can use it to describe the other parameters. From (4.18) and (4.19) we can obtain an expression for the ratio between the particle's rest mass in the Lorentzian region to its rest mass in the Euclidean region:

$$\left(\frac{m_l}{m_e}\right)^2 = \frac{1 - 2M/R}{2M/R - 1 + 2h_l^2} \quad (4.20)$$

From (4.20) we can see that the absolute value of the particle's Euclidean mass is always greater than it's Lorentzian mass. Correspondingly, from (4.18) we can observe that $h_l^2 \geq h_e^2$.

The paths of most interest to us are either an incoming Lorentzian geodesic matched to an incoming Euclidean one, or an outgoing Euclidean geodesic matched to an outgoing Lorentzian one. In other words we are looking for those transitions in which the Euclidean region describes the behavior of the geodesic at its minimum (those transitions are represented in figure 4.4, by sketches (a) and (b)). In figure 4.5 we consider the geodesics passing at a particular R value when we introduce the signature change. The horizontal axis is the h_l^2 parameter and the plot covers a representative range of permissible h_l^2 values for the geodesics passing through that R value. Vertical slices of the graph correspond to any one geodesic, and depict the maximum radius of that geodesic path (when it exists, $h_l < 1$) and the minimum radius reached by that same geodesic in the Euclidean region. On the same plot we include also the particle's rest mass ratio squared $((m_l/m_e)^2)$ for each particular combined geodesic, and its h_e^2 value.

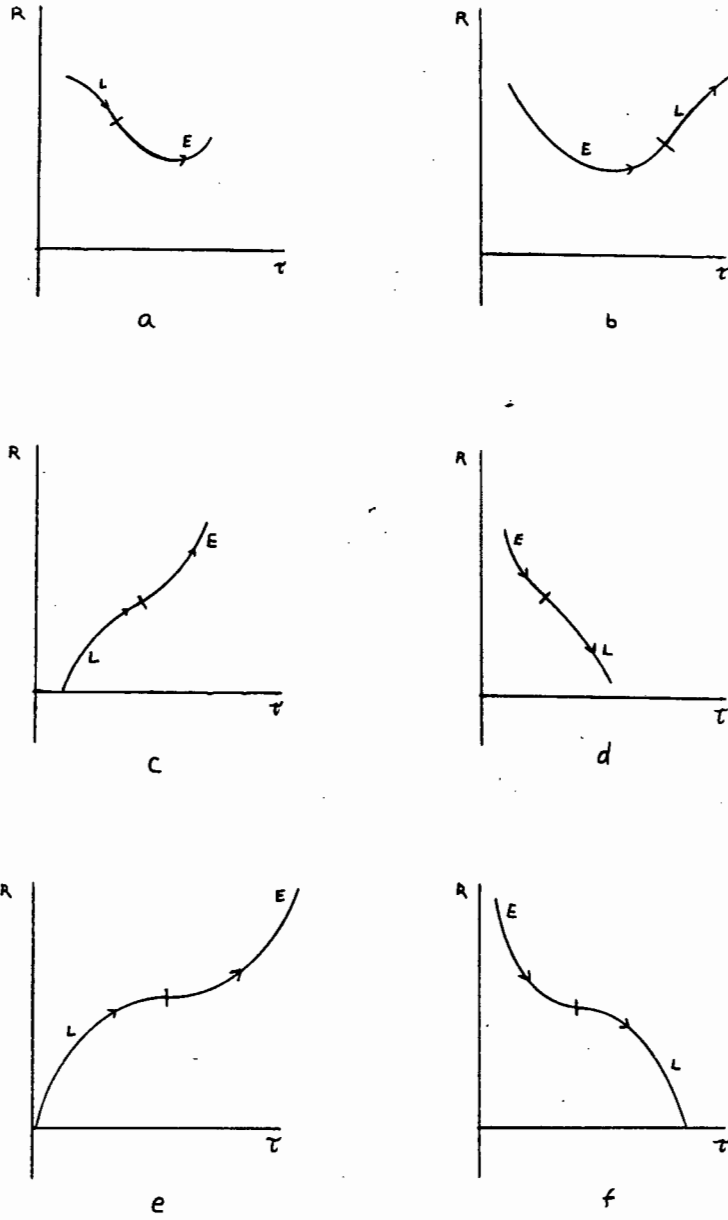


Figure 4.4: Schematic diagram illustrating the matching possibilities of geodesics across the transition. Sketches a & b are the desired transitions which lead to a bounce. Sketches c, d, e & f are of transition in which R goes to zero, e& f describing the case of matching at maximum and minimum expansion of the Lorentzian and Euclidean regions respectively.

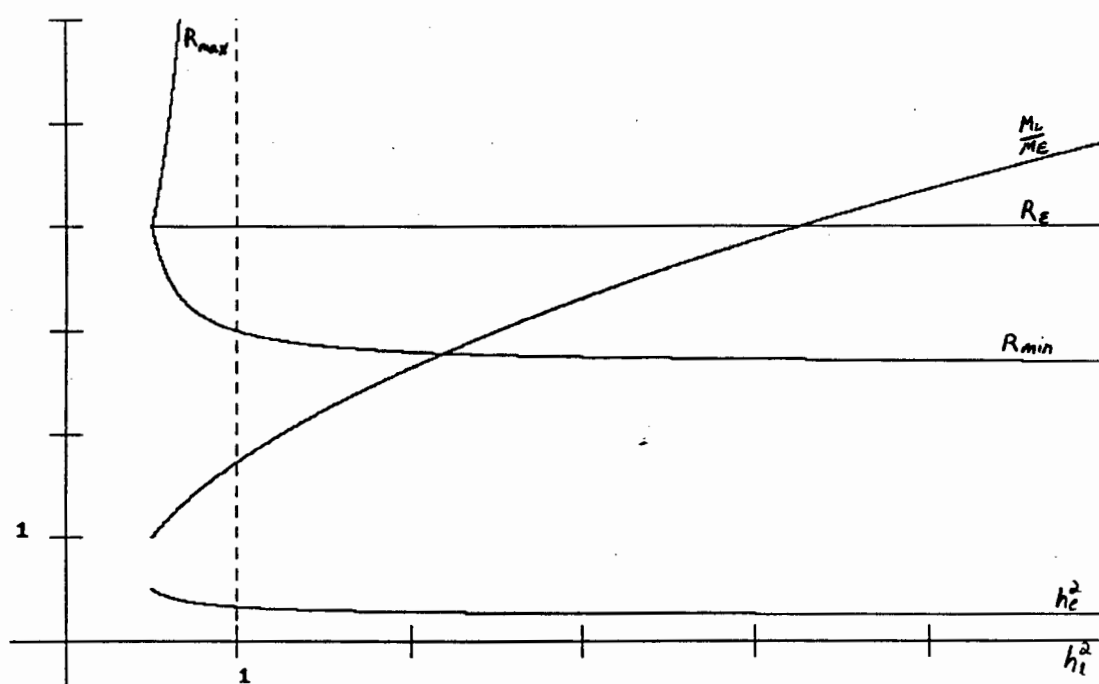


Figure 4.5: Diagram illustrating properties of combined Euclidean and Lorentzian geodesic paths.

Constant R surface

In this case conditions (4.17) lead to the following relations:

$$h_l m_l = h_e m_e \quad (4.21)$$

$$m_l \sqrt{2M/R - 1 + h_l^2} = m_e \sqrt{2M/R - 1 - h_e^2} \quad (4.22)$$

Similarly in this case, conditions (4.21) and (4.22) can be easily satisfied.

Chapter 5

The Tolman case

We now shift our attention to the Tolman metric. Primarily we do so because, in the vacuum case, the Tolman metric with appropriate choice of parameters can describe the full Schwarzschild-Kruskal- Szekeres manifold. The further advantage gained by the use of the Tolman metric is that we avoid the coordinate singularity at $R = 2M$ and the accompanying change of character of the Schwarzschild R and T coordinates. This now allows us to treat the entire space, and not only sections of it as was done in the Schwarzschild case. Also it is clear which metric element should change sign at a change of signature. In this chapter we first construct the evolution equations for this metric. We then develop the junction conditions and finally attempt to match regions with different signature. This is done quite generally so that we have the additional option of considering non-empty models.

5.1 Constructing the Solution

We start with a diagonal, synchronous, spherically symmetric metric, with an added factor of n , $n = \pm 1$.

$$ds^2 = -ndt^2 + B^2(t, r)dr^2 + R^2(t, r)d\Omega^2 \quad (5.1)$$

this leads to the following Einstein tensor :

$$G_{00} = \frac{2B'R'nR + 2\dot{B}\dot{R}B^2R - 2R''BRn - R'^2Bn + \dot{R}^2B^3 + B^3n}{B^3R^2} \quad (5.2)$$

$$G_{01} = \frac{2(\dot{B}R' - \dot{R}'B)}{BR} \quad (5.3)$$

$$G_{11} = \frac{(R')^2n - 2\ddot{R}B^2R - \dot{R}^2B^2 - B^2n}{nR^2} \quad (5.4)$$

$$G_{22} = \frac{R(-B'R'n - \ddot{B}B^2R - \dot{B}\dot{R}B^2 + R''Bn - \ddot{R}B^3)}{B^3n} \quad (5.5)$$

$$G_{33} = \sin^2\theta G_{22} \quad (5.6)$$

where $' \equiv \partial/\partial r$ & $\dot{} \equiv \partial/\partial t$ and the cosmological constant is taken to be zero.

We now follow the construction of the evolution equation similarly to [3]. We are assuming a perfect, comoving fluid so $G_{01} = 0$ which gives us the constraint

$$B = \frac{R'}{W(r)}, \quad W \geq 0 \quad (5.7)$$

where $W(r)$ is an arbitrary function of r . The metric now takes the standard Tolman form. Substituting (5.7) into the EFEs and defining the function

$$U \equiv R(1 + \dot{R}^2/n - W^2) \quad (5.8)$$

the Einstein tensor becomes :

$$G^0_0 = \frac{-U'}{R^2R'} \quad , \quad G^1_1 = -\frac{\dot{U}}{R^2\dot{R}} \quad , \quad G^2_2 = -\frac{1}{2RR'}\frac{d}{dr}\left(\dot{U}/\dot{R}\right)$$

Restricting our attention to a pressure free energy stress tensor, i.e. $G_{11} = G_{22} = G_{33} = 0$, gives the following expression for U :

$$U = S(\tau) \quad (5.9)$$

where $S(\tau)$ is an arbitrary function of τ . Equating expression (5.9) for U with its defining equation (5.8) leads to the evolution equation

$$\dot{R}^2 = (S/R + f) n, \quad f = W^2 - 1 \quad (5.10)$$

or

$$dt = \left[\left(\frac{S}{R} + f \right) n \right]^{-\frac{1}{2}} dR$$

In the standard ($n = +1$) Tolman evolution equation, the function $f(\tau)$ is a kind of local energy constant which determines the type of time evolution, viz: elliptic, parabolic or hyperbolic. The function $S(\tau)/2$ represents the effective gravitational mass within radius τ (hence from now on we write $M(\tau) \equiv S(\tau)/2$), while the density is given by

$$8\pi\rho = \frac{2M'(\tau)}{R^2 R'} \quad (5.11)$$

The kretschmann scalar for the metric (5.1) is:

$$\begin{aligned} k &= 4 \left(\frac{\ddot{R}^2}{R'^2} + \frac{2\ddot{R}^2}{R^2} + \frac{(2\dot{R}\dot{R}' - nf')^2}{2R'^2 R^2} + \frac{(\dot{R}^2 - nf)^2}{R^4} \right) \\ &= 4 \left(\frac{3M'^2}{R^4 R'^2} - \frac{8M'M}{R^5 R'} + \frac{12M^2}{R^6} \right) \end{aligned} \quad (5.12)$$

A quick analysis of equation (5.10) reveals a great deal of information on the nature of this space. The acceleration of the areal radius is

$$\ddot{R} = -\frac{nM}{R^2} \quad (5.13)$$

In the Euclidean case ($n = -1$) this reduces to $\ddot{R} = M_e/(R^2)$ showing that a bouncing Euclidean universe can be achieved if we select a positive “mass” term for the Euclidean region. Choosing M_e positive requires f_e to be negative in order to keep the \dot{R} term real.

5.2 Solutions of Evolution equations

We now concentrate on obtaining specific solutions to the differential equation (5.10). It is convenient to solve the evolution equations parametrically. We express both $R(r, t)$ and t in terms of the parameter η . We start with the solutions in the Lorentzian region where $n = 1$. First the elliptic solution

$$f_l < 0 : \quad R(t, r) = \frac{M_l}{|f_l|} (1 - \cos \eta_l), \quad t = \frac{M_l}{|f_l|^{\frac{3}{2}}} (\eta_l - \sin \eta_l) + a_l(r) \quad (5.14)$$

The parabolic solution

$$f_l = 0 : \quad t = 2/3 (2M_l)^{-\frac{1}{2}} R(t, r)^{\frac{3}{2}} + a_l(r) \quad (5.15)$$

The hyperbolic solution

$$f_l > 0 : \quad R(t, r) = \frac{M_l}{f_l} (\cosh \eta_l - 1) , \quad t = \frac{2M_l}{2f_l^{\frac{3}{2}}} (\sinh \eta_l - \eta_l) + a_l(r) \quad (5.16)$$

When $M_l = 0$ the equations change character and we get a linear solution: (Minkowski space):

$$f_l > 0 : \quad t = \frac{R(t, r)}{\sqrt{f_l}} + a_l(r) \quad (5.17)$$

a_l is an arbitrary function of r , and in the standard Tolman metric is interpreted

as the time of the big bang, or if we use the time reverse of the above equations, the big crunch time.

We now turn to the Euclidean solutions , $n = -1$. In this case there are more solutions since we can't be so sure we must disregard solutions with a negative "mass" term in Euclidean signature physics. We start with solutions with positive mass $M_e > 0$:

$$f_e < 0 : R(t, r) = \frac{M_e}{|f_e|} (\cosh \eta_e + 1) , \quad t = \frac{M_e}{|f_e|^{\frac{3}{2}}} (\sinh \eta_e + \eta_e) + a_e(r) \quad (5.18)$$

There are no solutions for $f_e = 0$ and $f_e > 0$

Solution with zero mass:

$$M_e = 0 : \quad f_e < 0 \quad t = \frac{R(t, r)}{\sqrt{|f_e|}} + a_e(r) \quad (5.19)$$

Now for solutions with negative mass $M_e < 0$:

$$f_e < 0 : R(t, r) = \frac{|M_e|}{|f_e|} (\cosh \eta_e - 1), \quad t = \frac{|M_e|}{|f_e|^{\frac{3}{2}}} (\sinh \eta_e - \eta_e) + a_e(r)$$

$$f_e = 0 : t = 2/3 (|2M_e|)^{-\frac{1}{2}} R(t, r)^{\frac{3}{2}} + a_e(r)$$

$$f_e > 0 : R(t, r) = \frac{|M_e|}{f_e} (1 - \cos \eta_e), \quad t = \frac{|M_e|}{f_e^{\frac{3}{2}}} (\eta_e - \sin \eta_e) + a_e(r)$$

One common feature in the evolution of all the negative "mass" models, is the fact that they all reach a singularity at zero areal radius at some point of their evolution. This supports our earlier assertion that these models can not give rise to a bounce.

One aspect of using synchronous comoving coordinates in a dust model is that the coordinates themselves describe geodesic paths. This means that the

coordinates are comoving with matter, when present, and the evolution equations derived above then describe the paths of the particles of matter. From (5.11) we can see that the density diverges at $R' = 0$. This occurs as a result of the matter shells crossing. Shells of matter at a different constant r , can intersect each other when they correspond to the same value of $R(r, t)$, which determines their areal radius evolution. In the vacuum case this does not present a physical problem since the density is zero, but it does introduce a bad coordinate coverage of the space. In non-vacuum cases care needs to be taken to select evolution equations which do not give rise to these physically troublesome shell crossings.

Any Tolman model with $M' = 0$ is a vacuum model (5.11), and thus represents at least a section of the Kruskal-Szekers-Schwarzschild space time, but not every selection of the arbitrary functions guarantees that we cover the entire manifold. The Novikov coordinates [6] do cover the entire manifold, and are obtained with the following choices for $a(r)$ and $f(r)$ [16]. To make the big bang and the big crunch a reflection of each other about $t = 0$ we can set

$$a_l(r) = \frac{-\pi M}{(-f)^{3/2}} \quad (5.20)$$

which means that the surface $t = 0$ is a simultaneous time of maximum expansion. $f(r)$ has a range from -1 to 0, where $f(0) = -1$ at the Schwarzschild throat, and it increases monotonically to 0 as $r \rightarrow \pm\infty$. Novikov's choice for this function is

$$f_N = \frac{-1}{1 + \left(\frac{r}{2M}\right)^2}$$

5.3 Matching conditions

We perform the matching on the simplest possible surface, the constant time surface $t = \text{constant}$. This is merely a coordinate restriction and not a physical one, because the origin of the time coordinate, $a(r)$, is an arbitrary function of position. It amounts to finding the family of geodesics orthogonal to the transition surface, and using these as lines of constant r . The intrinsic metric of such a surface is correspondingly simple:

$$d\sigma^2 = B^2 dr^2 + R^2 d\Omega^2 \quad (5.21)$$

When matching, a reasonable choice is to equate the angular parts $d\Omega_e = d\Omega_l$ which fixes $R_e = R_l = R_\Sigma$. We are free to rescale the radial coordinate on either side, so we also set $r_e = r_l$, which fixes $B_e^2 = B_l^2 = B_\Sigma^2$. Since R is continuous across the junction and is a function of r on the junction surface ($t = \text{const}$), i.e. $R_\Sigma = R_\Sigma(r)$, we have also that $R'_e = R'_l$. Inserting the condition (5.7) we obtain

$$\left(\frac{R'_e}{W_e}\right)^2 = \left(\frac{R'_l}{W_l}\right)^2 \quad (5.22)$$

leading to $W_e^2 = W_l^2$ and hence $f_e = f_l = f$.

The non-zero elements of the extrinsic curvature are:

$$K_{rr} = \frac{-\sqrt{\epsilon_3} \dot{B} B}{\sqrt{-n}} \quad , \quad K_{\theta\theta} = \frac{-\sqrt{\epsilon_3} \dot{R} R}{\sqrt{-n}} \quad , \quad K_{\phi\phi} = \sin^2 \theta K_{\theta\theta} \quad (5.23)$$

Note again that e_3 is the magnitude of the normals and in this case is given by $e_3 = -n$.

From the $K_{\theta\theta}$ terms we get the equivalence $\dot{R}_l = \dot{R}_e$ which again implies $\dot{R}'_l = \dot{R}'_e$. The latter also comes from matching K_{rr} . Imposing the matching conditions on the differential equation (5.10) we find the matching conditions for

the mass terms of the respective regions.

$$M_e - M_l = -\dot{R}_\Sigma^2 R_\Sigma \quad (5.24)$$

or

$$M_l + M_e = -f R_\Sigma \quad (5.25)$$

Thus we have a condition relating the mass terms of each region, at every point in the junction surface. To recap, we require that on the transition surface, given by $t_{e\Sigma} = \text{constant}$ and $t_{l\Sigma} = \text{constant}$, the following hold everywhere on Σ :

$$R_e = R_l = R_\Sigma(r)$$

$$\dot{R}_e = \dot{R}_l = \dot{R}_\Sigma(r)$$

$$f_e = f_l = f(r)$$

$$M_e = -M_l - f R_\Sigma$$

The fulfillment of these relations ensures that the regions are matched.

5.4 Finding the junction surfaces

Using the matching conditions from (5.24) above we can now obtain solutions for the matching surfaces. We do so by inspecting the parametric solutions describing the evolution of the space and imposing the foregoing matching conditions. This establishes the locus of the junction surface in space and when it can occur in terms of the evolution of the two regions.

The matching of the tangent vectors to the coordinate geodesics is automatically fulfilled if we consider that the tangent vectors to these geodesics (comoving

with the particles of matter) posses no radial component, and the time components are trivially matched (both have a magnitude of one).

The principal feature we are looking for is a bouncing universe, meaning a Lorentzian region matched to a bouncing Euclidean region that in turn may be matched to another Lorentzian region. This involves establishing the existence of at least two solution surfaces in the Euclidean region of the model under investigation. In general, given two space-like hypersurfaces, there will not be any geodesics that are orthogonal to both, so requiring both to be $t = \text{constant}$ surfaces in the same coordinate system could be restrictive. For vacuum Euclidean regions, we get round this by considering coordinate transformations. This assumes a Euclidean equivalent of Birkoffs theorem.

One of our original aims of finding a transition surface completely enclosed within the event horizon is dealt an immediate blow. In the Novikov type vacuum models the function f has the range $[-1, 0]$ or $[-1, c] : c \geq 0$. From the constant time transition boundary condition (5.25), we obtain under the constraint $R < 2M$ that:

$$M_e \leq M_l (-1 - 2f)$$

This indicates that we can not have a solution surface contained inside the event horizon without the mass in the Euclidean side becoming negative in the outer regions, i.e. at large r . As pointed out earlier, a negative mass Euclidean solution does not possess the desired bounce properties. This is not unlike the result obtained in the Schwarzschild metric for the constant R surface. Actually it is apparent that in order to maintain M_e non negative we are restricted to the choice $-1 \leq f \leq -1/2$.

We have five functions which are as yet unspecified $f(r)$, $M_e(r)$, $M_l(r)$, $a_l(r)$

and $a_e(r)$. In the following sections we find the allowed junction surfaces which are obtained for different selections of the arbitrary functions. The main function which we selected to vary between the different models is the mass function. After a description of general relations between the various functions on the surface, we study three case, all are transitions from a vacuum Lorentzian region to a Euclidean region with a varying degree of freedom in the selection of its mass function M_e .

General transitions

We attempt to derive several general relations between the different arbitrary functions which necessarily have to hold true on the surface of transition in order to satisfy the matching conditions. We consider the mass functions M_e and M_l to be as yet unrestricted functions of r . We consider models with negative f and positive Euclidean mass term, as this are the only models which can give rise to a Euclidean region with a bounce.

Inserting (5.24) into the differential equation (5.10) and solving for R we obtain the following expression for R on the transition surface:

$$R_\Sigma = \frac{-M_l - M_e}{f} \quad (5.26)$$

Equating this value of R_Σ with its parametric description in the Lorentzian case (5.14) yields

$$\begin{aligned} \frac{M_l}{|f|} (1 - \cos \eta_{l\Sigma}) &= \frac{-M_l - M_e}{f} \\ \Rightarrow \cos \eta_{l\Sigma} &= -\frac{M_e}{M_l} \end{aligned} \quad (5.27)$$

A similar analysis on the Euclidean side leads to

$$\cosh \eta_{e\Sigma} = \frac{M_l}{M_e} \quad (5.28)$$

The areal radius on either side of the surface is now automatically matched when using the above relations for the parameters η_e & η_l .

On the transition surface t is a constant, so we take $\eta_{e\Sigma}$ as a function of r only (also true for $\eta_{l\Sigma}$). This leads to the following relation between $\eta_{e\Sigma}$, M_e and f :

$$t_\Sigma = \frac{M_e}{|f|^{\frac{3}{2}}} (\sinh \eta_{e\Sigma} + \eta_{e\Sigma}) + a_e(r) \quad (5.29)$$

If possible, we choose $a_e(r) = 0$, this means that (5.29) will now give a symmetrical coordinate coverage in the Euclidean region. This permits a second copy of any transition surface found away from $t = 0$, and thus ensures a bounce.

A further restriction which we can impose, in order to reduce further the choice of solutions, is to require that the transition occur at the same time, t_Σ , for both regions, i.e. t is continuous through Σ . On equating the transition time we set the following form for the function a in the Lorentzian region

$$a_l(r) = t_\Sigma - \frac{M_l}{|f|^{3/2}} (\eta_{l\Sigma} - \sin(\eta_{l\Sigma})) \quad (5.30)$$

To obtain a specific solution we can fix any two of M_l, M_e, a_l, a_e , to obtain the others.

We include a schematic diagram 5.1 of the evolution of the areal radius R in terms of the evolution parameter η , to illustrate where the transition can take place in terms of conditions (5.27) and (5.28). Furthermore if we insist on matching the transition time, we have the following expressions for the evolution parameter on the transition surface:

$$\begin{aligned} t_\Sigma > 0 & : \quad \eta_{l\Sigma} = \cos^{-1}(-M_e/M_l) \\ & : \quad \eta_{e\Sigma} = \cosh^{-1}(M_l/M_e) \end{aligned}$$

or

$$t_\Sigma < 0 : \quad \eta_{l\Sigma} = 2\pi - \cos^{-1}(-M_e/M_l) \quad (5.31)$$

$$: \quad \eta_{e\Sigma} = -\cosh^{-1}(M_l/M_e) \quad (5.32)$$

The above results apply to signature change on a constant t surface in any Tolman model, empty or not.

Vacuum to vacuum equivalent mass transition

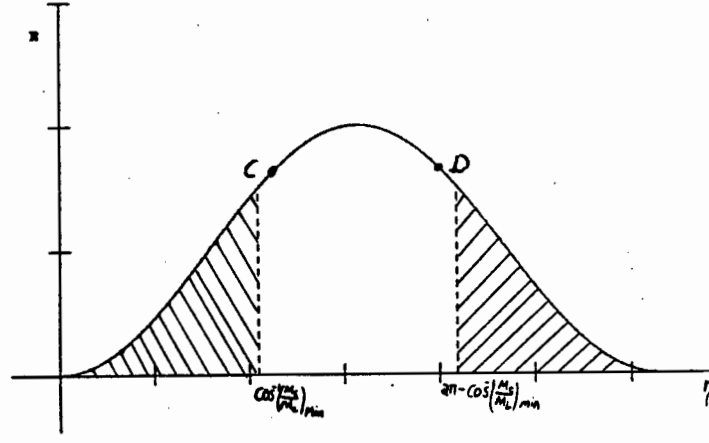
We set the mass term to a constant (hence $\rho = 0$) on either side and then equate the mass terms of both regions $M_l = M_e$. This setting gives us from (5.24) that $\dot{R}_\Sigma = 0$ on the matching surface. The areal radius is given, in the Lorentzian region by

$$R_\Sigma = \frac{2M}{2|f|} (1 - \cos \eta_{l\Sigma})$$

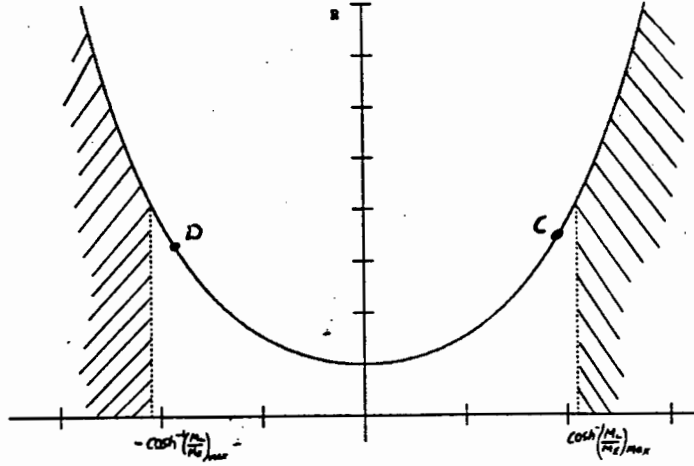
and in the Euclidean region by:

$$R_\Sigma = \frac{2M}{2|f|} (\cosh \eta_{e\Sigma} + 1)$$

Matching the areal radius is possible and can occur only at the point $\eta_{e\Sigma} = 0$, $\eta_{l\Sigma} = \pi$, the locus of minimum and maximum expansion of the respective sets



a



b

Figure 5.1: A schematic diagram of areal radius evolution a) In the Lorentzian region b) In the Euclidean region. The shaded region indicates evolution stages at which no transition can occur. Point C indicates the regions in which the matching occurs for $\dot{R} > 0$, and point D indicates matching at $\dot{R} < 0$.

of coordinates. We can now observe that this case is equivalent to a constant T transition in the Schwarzschild case. In both cases we have a Schwarzschild space on the Lorentzian side, and the mass terms of both regions are equal. The matching surface is given in this case by:

$$R_{\Sigma} = \frac{2M}{|f|} \quad , \quad 0 < |f| \leq 1$$

This matching surface (like the one in the Schwarzschild case), touches $R = 2M$ but otherwise lies entirely outside the horizon. Also, all Schwarzschild constant T surfaces intersect the neck at its moment of maximum expansion. Thus this case has both $T = \text{constant}$ and $t = \text{constant}$. If we set $a_e = 0$, to obtain a symmetric coverage of the Euclidean region, we have that the transition time is $t_{\Sigma} = 0$. From (5.30) we can see that the resulting a_l also gives us symmetric coverage of the Lorentzian region about $t = 0$ (5.20).

Vacuum to vacuum

For vacuum regions the mass term M_e needs to be constant, but in this case we do not insist on the equality of the masses on either side. From (5.27) and (5.28) we observe that both η_e and η_l are constant on the transition surface. This means that we can now use the parametric description for the t coordinate, (5.18), to establish the relation between the two arbitrary functions of $f(r)$ and $a(r)$. If we were to follow our extra restriction requiring a symmetric coverage of the Euclidean region (5.29) then we would find that we cannot have a constant time transition surface (since f is non-constant and it is the only function in the expression). In order to be able to have a transition surface in this model we have to abandon that time symmetry condition, and use the general time equation (5.29).

Once fixed, the $a_e(r)$ function remains unchanged in that choice of the coordinates (i.e. off the transition surface). For given M_e and M_l this gives us only one possible solution. Hence we do not have an assured second transition at the time reversal any more. A symmetric coverage assuring the existence of another transition surface, is however no longer a necessary requirement. This is so since we are dealing with a vacuum case. In the vacuum case, all coordinate systems are equivalent to each other. We can transform the coordinate system arrived at from one matching, to another coordinate system which fulfills the matching conditions on another transition surface.

Lorentzian vacuum to Euclidean non-vacuum

In this model we set $M_l = \text{constant}$ and $M_e = M_e(r)$. In order to obtain a transition surface, we plot the various variables numerically as a function of a parameter, selecting the mass function M_e as the parameter with which to describe the other variables on the junction surface.

Since the Euclidean region is no longer empty, not all geodesic coordinate systems are equivalent to the comoving one. We thus desire a symmetric coordinate coverage in the Euclidean region to ensure the existence of a second transition. Starting with the time symmetric time evolution equation of the Euclidean region

$$t = \frac{M_e}{|f|^{\frac{3}{2}}} (\sinh \eta_e + \eta_e) \quad (5.33)$$

From (5.29) (with $a_e = 0$) and (5.33) we obtain:

$$F(r) \equiv |f|^{\frac{3}{2}} \frac{t}{M_l} = \frac{\sinh \eta_e \Sigma + \eta_e \Sigma}{\cosh \eta_e \Sigma} \equiv D(\eta_e \Sigma)$$

Where the right hand side defines $D(\eta_e \Sigma)$, and the left hand side defines $F(r)$. We

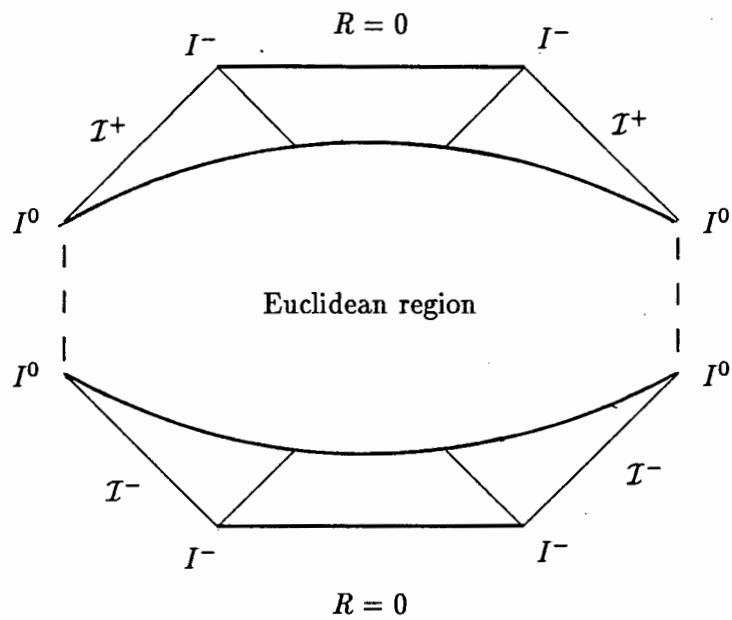


Figure 5.2: Constant t surface transition, $M_e \neq M_l$. The top and bottom regions are Penrose diagrams for the Schwarzschild-Kruskal-Szekeres manifold. The central region doesn't have a causal structure.

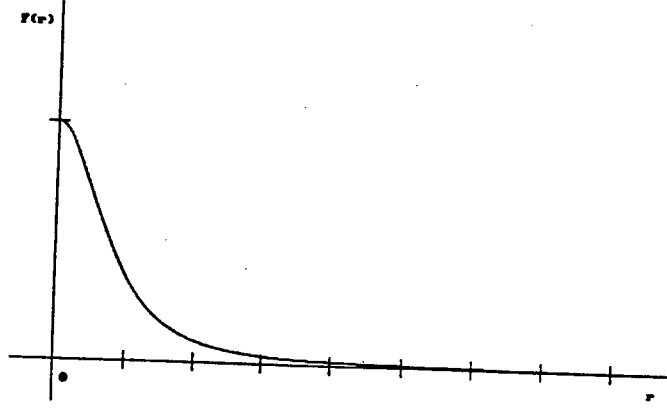


Figure 5.3: $F(r)$ vs. r

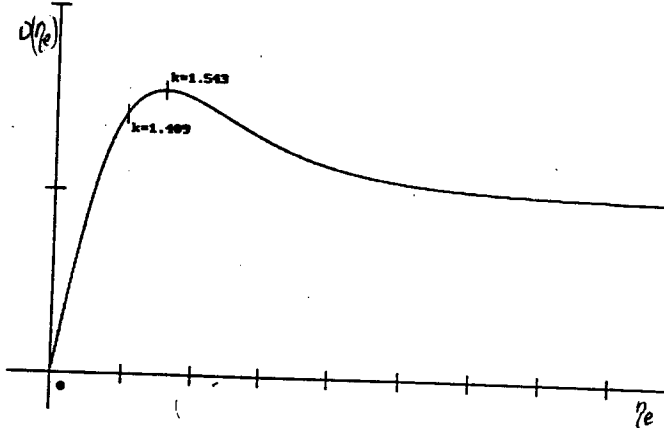


Figure 5.4: $D(\eta_e \Sigma)$ vs. $\eta_e \Sigma$.

select for this numerical case that the function f to take the Novikov form,

$$f = \frac{-1}{1 + r^2} \quad (5.34)$$

The F function with choice (5.34) for f , now needs be matched to the D function above in order to fix the r dependence of $\eta_{e\Sigma}(r)$. We now plot both functions in figures 5.3,5.4 to illustrate the matching needed.

The function f spans the values -1 to 0 monotonically, and we have at our disposal only one constant ($t_\Sigma/M_l \equiv k$) which is freely adjustable, so we need to

select the section of the D graph which includes 0. Also since f is monotonic, the matching of D and f cannot be achieved for a range of $\eta_{e\Sigma}$ including $\eta_{e\Sigma}$ values greater than the one at the maximum of D . Hence we have a restriction on when in the Euclidean evolution the transition can occur.

Another condition on this matching is that we must have a complete coordinate coverage of the space, i.e r should span all values $-\infty$ to $+\infty$. In particular it must include the neck at $r = 0$, $f = -1$. To effect this we set $f = -1$ and obtain a restriction on the constant of proportionality in terms of $\eta_{e\Sigma}$.

$$f = -1 \Rightarrow \frac{t_\Sigma}{M_l} = \frac{\sinh \eta_{e\Sigma} + \eta_{e\Sigma}}{\cosh \eta_{e\Sigma}} \quad (5.35)$$

This relationship sets the value of $\eta_{e\Sigma}$ at $r = 0$ (i.e. $\max \eta_{e\Sigma}$). The allowed range for k (i.e. possible positioning of the transition surface) is from 0 to the maximum value of $D(\eta_{e\Sigma})$, $D_{\max}(\eta_{e\Sigma}) = 1.543$, which occurs at

$$\eta_{e\Sigma} \sinh \eta_{e\Sigma} - \cosh \eta_{e\Sigma} = 1 \rightarrow \eta_{e\Sigma} = 1.543$$

Once we select a value, say $k|_0$ corresponding to $r = 0$ (i.e choose the location of the transition surface), the values permitted for $\eta_{e\Sigma}$ are the ones for which $D(\eta_{e\Sigma})$ runs from $k|_0$ ($r = 0$) to $D(\eta_{e\Sigma}) = 0$, corresponding to $r = \pm\infty$.

From the discussion above concerning (5.35), we can observe from (5.35) and (5.28) that any particular choice for k restricts the allowed range of permissible M_e values on the surface. The allowed range for the euclidean “mass” term given a Lorentzian mass M_l is

$$M_l / \cosh \eta_{e\Sigma} \leq M_e \leq M_l \quad (5.36)$$

where the $\eta_{e\Sigma}$ value in (5.36) is the solution of (5.35) for the chosen value of k .

Our plotting procedure is as follows:

- Select a Lorentzian mass term M_l .
- Choose a transition time t which complies with (5.35). This fixes k .
- Generate values of M_e which span all allowed values as permitted by (5.35) with our choices for t_Σ and M_l .
- For each M_e value calculate the following values:
 - (i) f from (5.33) and r from the Novikov form (5.34).
 - (ii) R from $R = \frac{M_l + M_e}{f}$.
 - (iii) a_l from (5.30).
 - (iv) density ρ from (5.11) (calculating ρ requires finding the values of R^2 , R' and M' . R^2 is obtained from (5.26), the values of M'_e , R' are obtained from the derivatives with respect to r of (5.29) & (5.28), (5.26) respectively).

We plot the values of R , a_l and ρ on the transition surface as functions of r . The values of both R and ρ remain the same whether the transition occurs at a positive or a negative time, but the values of a_l differ in the two cases. However the function a_l for a positive time transition is only an apparent bang time since the stage of the Lorentzian evolution for which it would have given the bang time is described by a Euclidean region. Thus only the values of a_l for a negative time transition, have a physical meaning.

The three graphs are plotted for two different values of k . Figures 5.5, 5.7 and 5.9 correspond to the largest range possible for M_e (i.e. $\eta_{e\Sigma}$ running from 0 to its value at the maximum of $D(\eta_{e\Sigma})$, $k = 1543$. For comparison, in figures 5.6, 5.8 and

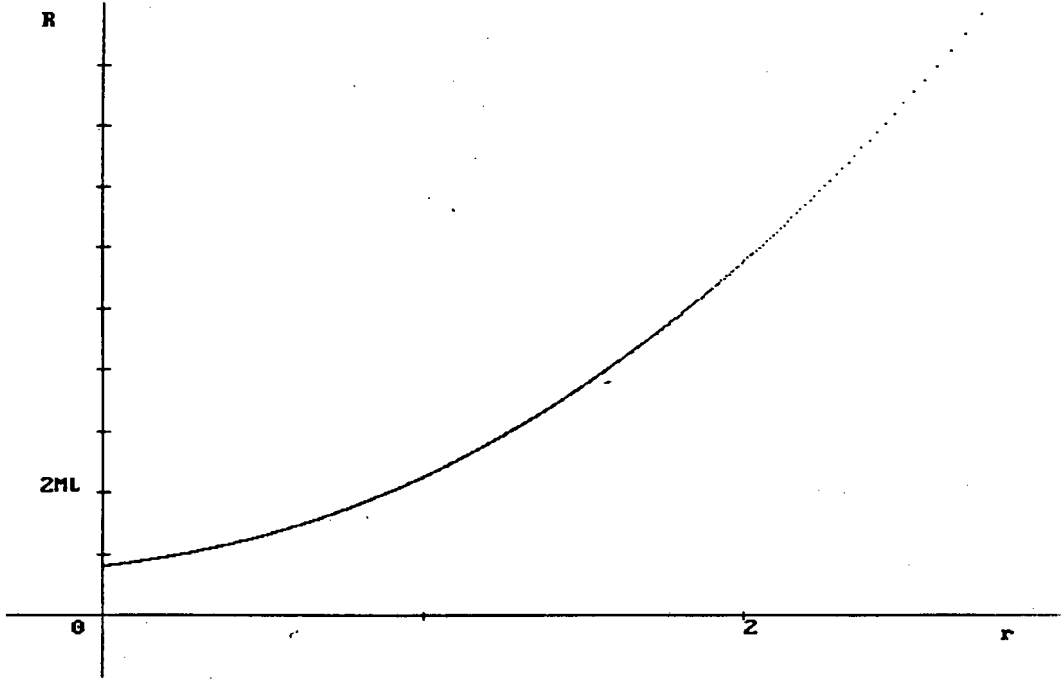


Figure 5.5: Aerial radius R vs. coordinate radius r , for $k = 1.543$, i.e. $\max k$. Note that R' is discontinuous at $r = 0$.

5.10 we selected a lower cutoff for $\eta_{e\Sigma}$ hence a smaller range of M_e , corresponding to a choice of $k = 1.409$.

Observing the graphs 5.5 and 5.6 we note that as expected the areal radius has a minimum and does not go singular. An interesting feature can be noted from 5.7, in this case, with $k = 1.543$, the bang time, a_l , for $t_\Sigma < 0$ has a maximum away from $r = 0$. Paper [16] indicates that if one has $R' > 0$ and $f < 0$ one requires $a' \leq 0$ to avoid shell crossing. In our case a' is positive for small r values and so shell crossings occur, but since we have a vacuum in the Lorentzian region this does not introduce physical problems. Figures 5.9, 5.10 show that the “density” is well behaved (does not diverge) on the surface, this indicates that the shell crossings in the Lorentzian region do not occur on the surface of transition.

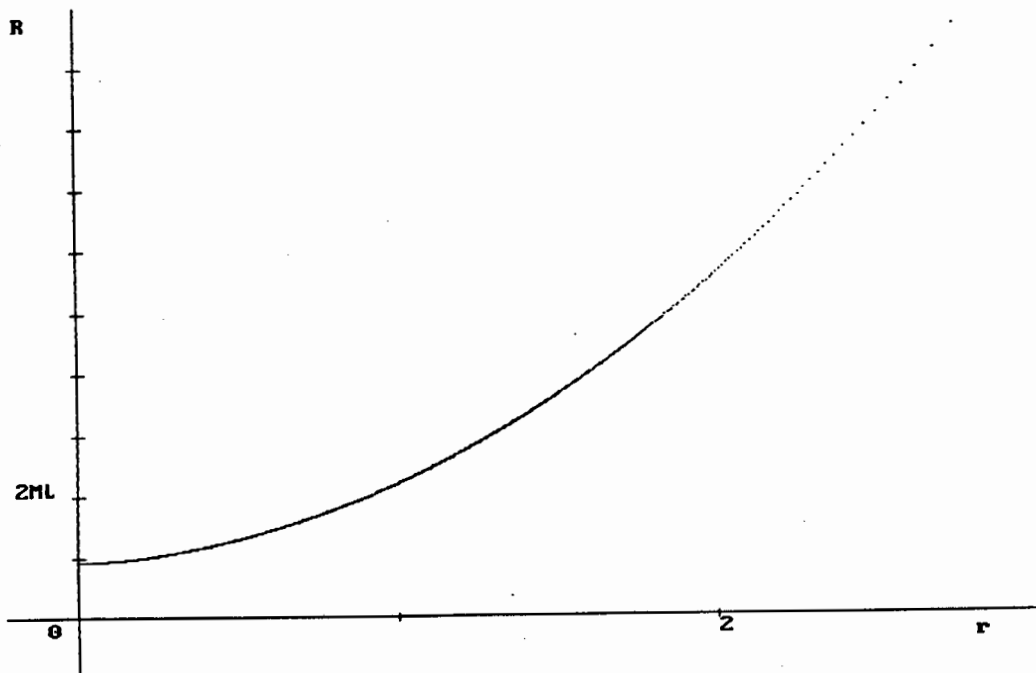


Figure 5.6: Aerial radius R vs. coordinate radius r , for $k = 1.409$

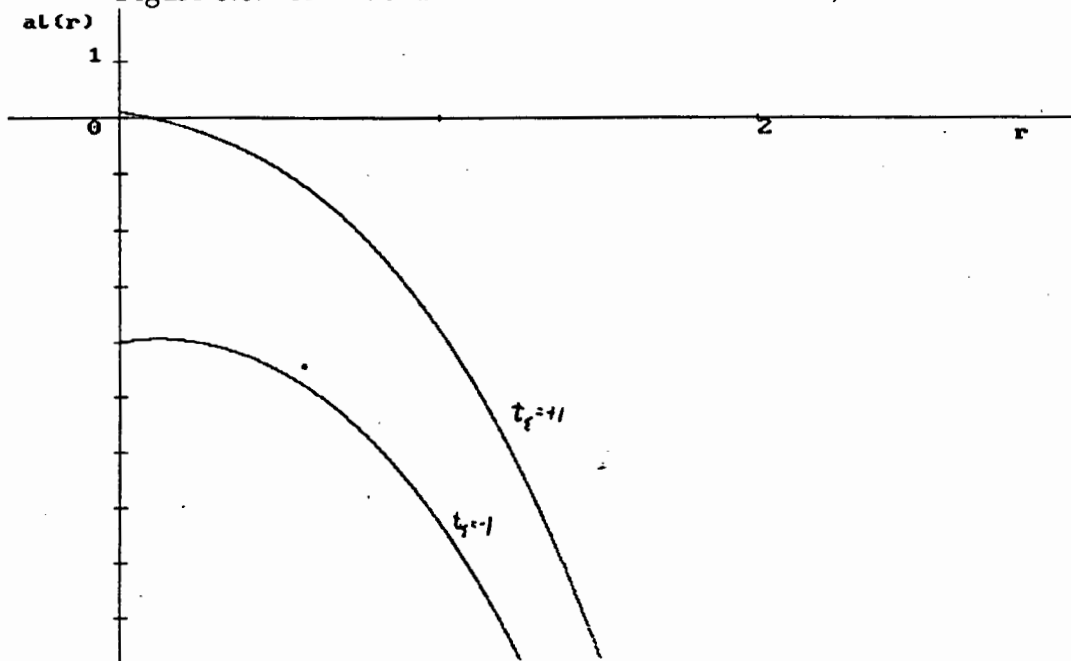


Figure 5.7: $a_l(r)$ vs r , for $k = 1.543$, $t_\Sigma = +1$ and $t_\Sigma = -1$. Note that a_l for $t_\Sigma = -1$ has its maximum away from $r = 0$.

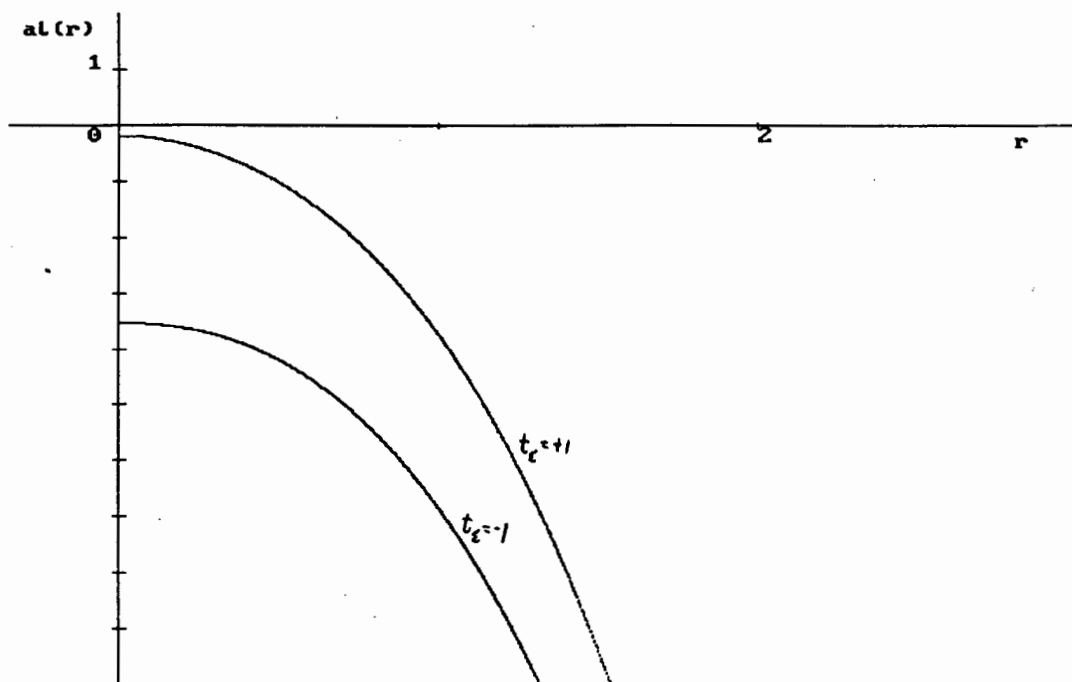


Figure 5.8: $a_l(r)$ vs r , for $k = 1.409$, $t_\Sigma = +1$ and $t_\Sigma = -1$. Note that a_l for $t_\Sigma = -1$ now has its maximum at $r = 0$.

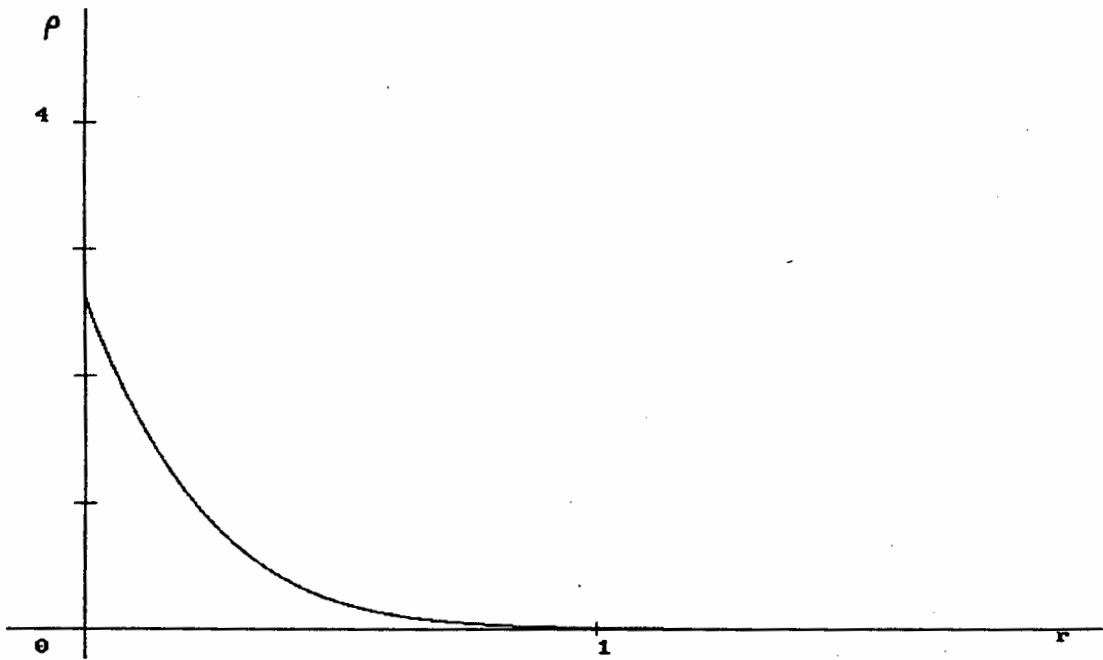


Figure 5.9: Density vs. r , for $k = 1.543$, i.e. $\max k$. Note that ρ' is discontinuous at $r = 0$, this follows from the discontinuity in figure 5.5.

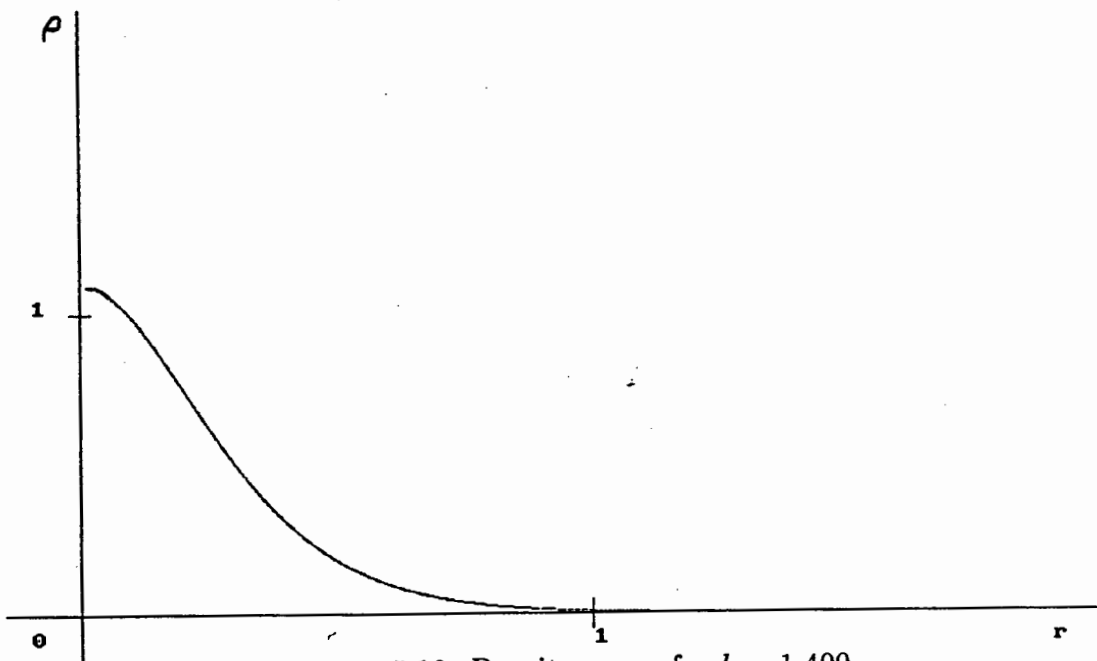


Figure 5.10: Density vs. r , for $k = 1.409$

Chapter 6

Conclusion

Before reviewing the results obtained in this thesis, we note the assumptions underlying the work. We have based our notion of manifold continuity on the fulfillment of the Darmois matching conditions. We observe that no modifications are necessary to the constant signature D conditions, when used in the context of signature change, provided the normal to the transition surface is properly defined. What is changed, is the notion of conservation through a signature transition, as shown through a re-analysis of Israel's identities and their relation to the divergence theorem. Another point to note is that the classical transitions constructed here, based on the Darmois matching as the minimal matching condition, may or may not be satisfactory from a quantum cosmological point of view.

Based on this selection for the matching conditions we managed to show that we can have signature transitions in a spherically symmetric vacuum space-time in both the Schwarzschild metric and the Tolman metric representations, though the ensuing Euclidean region might not be empty.

One of the initial aims of this classical approach was to see whether the singularity in space-time at $R = 0$ can be avoided with the introduction of classical Euclidean regions. We thus focused on the transition surfaces which are the boundaries of Euclidean regions which have the property that the geodesic paths bounce. In this way the singularity at $R = 0$ can be avoided. Within the Schwarzschild metric form, such a transition surface was possible on a constant Schwarzschild time T slice, but this can only span the outer region $R \geq 2M$. Conversely the constant R surface can be entirely inside the horizon, but does not lead to a bouncing Euclidean region. We were also able to find transitions which gave rise to a bounce within the Tolman metric form using constant Tolman time t transitions, occurring in an elliptic Lorentzian region. They include as a special case the equivalent of the Schwarzschild constant T transition, but also offer a greater variety of transitions. One of the added flexibilities offered in the Tolman transitions is that the matching surfaces can extend into the region $R < 2M$.

In the Tolman metric we were unable to find a transition surface which is fully enclosed within $R < 2M$ and bounded a Euclidean region with a bounce, but we have not ruled out the possibility that such surfaces can not be found in principal. This could be established through a natural extension of this research into the investigation of general transition surfaces in the Tolman metric. Another promising avenue is to extend the research by the use of the Kantowski-Sachs metric [21]. We are led into that avenue if we impose that the surface of transition occur at a constant R . This necessarily requires that f be constant on the transition and also that $R' = 0$, in order to maintain the transition 3-surface we are forced to the choice $f = -1$ for all r , otherwise our transition surface degenerates into a 2-surface. The resulting metric is no longer the Tolman metric but is the Kantowski-Sachs metric. The vacuum Kantowski-Sachs metric describes the

region inside the horizon of the Schwarzschild space-time with constant t surfaces that are also constant R , and correspondingly when using this metric we are restricted to the Schwarzschild region $R < 2M$. It would be interesting to investigate the limiting behavior of the Tolman metric for $R' = 0$, $f = -1$ to see whether it gives the Kantowski-Sachs metric.

There is nothing stopping us from having arbitrarily many transitions between Lorentzian and Euclidean regions. In each we set the one transition surface at t_Σ and the other transition surface symmetrically placed in the Euclidean space, at $-t_\Sigma$. Transforming the coordinates in the vacuum Lorentzian region allows us to have transitions at different choices for t_Σ . In the Schwarzschild case the spaces are static hence the choice of transition time is purely arbitrary.

The transitions we have modeled in the Tolman case can be used in Smolin's idea as a mechanism for universe creation, the neighborhood of the singularity at the end of the universe is replaced with a Euclidean region which in turn connects to the beginning of another Lorentzian universe. Transition surfaces fully enclosed within $R < 2M$ would have enabled us to generate universes originating at black holes, and thus provided the main mechanism for the creation of the child universes.

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Appendix A

Change of signature in classical Relativity

Change of signature in classical relativity

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February 1992

Abstract

We point out that the classical Einstein field equations, suitably interpreted, allow a change of signature of space-time. Specific examples of such changes are constructed in the case of Robertson-Walker geometries. We obtain classical solutions that have properties similar to those obtained in quantum cosmologies obeying the Hartle-Hawking 'no boundary' condition: these singularity-free universes have no beginning, but they do have an origin of time. They can be regarded either as classical analogues of the quantum cosmology results, or as classical solutions where a quantum cosmology era is avoided.

1 Introduction

One of the intriguing aspects of the Hartle-Hawking programme [1,2] investigating the wave-function of the universe is the idea that the signature of space-time should change at very early times (when quantum gravity effects dominate), resulting in an origin of the universe in a regime where there is no time (the space-time metric is positive definite, so that “space-time” is in fact purely spatial) and possibly there is no boundary (“space-time” in the spatial region is like a 4-sphere). Much has been made of the possible philosophical implications of this idea, e.g. by Stephen Hawking in *A Brief History of Time* [3] (for other non-technical accounts of the proposal, see [4]).

An interesting question that arises, then, is whether similar processes are possible in classical solutions of the Einstein field equations. At first one’s reaction is certainly not, all solutions maintain the same signature. However this is true in the usual solutions not because it is demanded by the field equations, but rather because it is a condition we normally impose on the metric before we start looking for solutions. The issue raised in this paper is, can we find solutions of the classical Einstein field equations where we do not make this assumption, leading to situations where the signature does indeed change sign (as envisaged in the quantum solutions)?

There is a problem here of some significance: namely in order for the signature S of the metric g_{ab} to change sign, it must go through the value

zero; at those events where $S = 0$, the space-time metric is singular. Does this not destroy the project before we begin, a fundamental assumption of General Relativity Theory being that the signature S is always non-zero?

This is one of those situations where it makes a difference what precise formulation we use for General Relativity. The standpoint adopted here will be that (in consonance with quantum cosmology) we essentially adopt a Hamiltonian approach (see e.g. [5-7]), based on a slicing of spacetime into space-sections, with lapse and shift functions indicating how the 3-spaces are stacked together (using a continuous time-coordinate to label these space sections) to form a 4-dimensional space-time. In this approach the metric of the 3-spaces, together with the appropriate matter variables, are regarded as representing the physical degrees of freedom. It is well known that on the one hand, the lapse function can be chosen arbitrarily (there is no Einstein equation that specifies its evolution), and on the other, as emphasized by Teitelboim [7], the Hamiltonian approach does not determine the signature of the space-time. Here we generalise the approach by removing the restriction that the signature has a definite sign; this is done by replacing what is normally written as N^2 by N . There is no problem in doing so; indeed it is noteworthy that in the usual Hamiltonian approach the choice of this variable as a squared function is unnatural; it is $N(t)$, not $N^2(t)$, that occurs naturally in the overall Hamiltonian as a Lagrange multiplier. The arbitrariness of $N(t)$ allows us to choose it to change sign, thereby causing the space-time signature to change. A form of singularity arises on the surface of change,

but it is a mild one which can be handled successfully provided one treats it with care.

To investigate the consequences of such a situation, we use a particular representation of the standard tensor form of the Einstein field equations (the same results can doubtless be obtained by suitable variational principles, without introducing a complex time coordinate, but investigating that issue is not our aim here). The burden of this paper then is that

we can find Robertson-Walker solutions of the classical Einstein field equations where the lapse function $N(t)$ changes sign at some time t_0 , but the matter density and pressure are finite and the 3-space metric $h_{\alpha\beta}$ is regular as the change of sign takes place.

Although the space-time apparently gets stuck at this surface when written in terms of a coordinate t for which $N(t)$ is continuous across the surface of change, this is a result of that specific coordinate choice (there is a form of coordinate singularity there). In this case $N \rightarrow 0$ as $t \rightarrow t_0$, but the proper time $s = \int \sqrt{|N(t)|} dt$ elapsed to that surface is finite and time measured by t speeds up indefinitely relative to proper time s as one approaches the surface Σ : $s' \equiv ds/dt = \sqrt{N(t)} \rightarrow 0$. Thus

$$t \neq t_0 \Rightarrow \frac{df}{dt} = \frac{df}{ds} \frac{ds}{dt} = \sqrt{|N|} \frac{df}{ds} \quad (1)$$

allows df/dt to have the limit zero on approaching Σ while df/ds has a non-zero limit there. For this reason we have to handle the transition across the

surface with the utmost care.

This space-time can be regarded as a classical solution where the metric changes signature but the geometry is - apart from this epochal feature - everywhere regular. The events where this change of signature happens represent (in a purely classical way) the transition from a classical regime with usual causal properties to a classical regime of a completely different nature, and so may for example be specified to take place at the Planck epoch in the Robertson-Walker universe, where quantum cosmology ideas suggest such a transition might take place. In this case we can use this classical solution as a model of the quantum cosmology predictions. However it is also interesting to contemplate the possibility that this change could happen at a less extreme stage, for example around the GUTs time; in this case we have the possibility of a purely classical theory that avoids the need for a quantum gravity epoch. In either case, we are able to give a precise criterion for when such a change should take place, if the matter content of space-time is a scalar field; essentially it occurs when the spatial curvature of the universe is equal to the potential energy of the scalar field. This criteria ensures that \dot{R}^2/R^2 remains positive, and this can be regarded as the reason the change of signature is required. Thus rather than being essentially an invocation of the idea of complex time, as one might at first think, the change of signature is introduced precisely to avoid the need for complex time in the extreme conditions that prevail in the early universe.

The evolution in the positive definite region is determined from the equation of state of the “matter” present there. It can for example have the feature desired by Hartle and Hawking [3,4] that this region is like a positive definite 4-sphere (without any singular point, that is, without boundary). Thus we can provide examples that agree with their concept of a space-time structure for the universe without boundary, but are based on a purely classical picture¹.

In the body of the paper, we give two ways of looking at this change of signature: first as a consequence of a continuous change in the lapse function $N(t)$, where the solution of the field equations is carried continuously through the epoch of signature change; and second, in terms of the discontinuous change of a signature function ϵ , that takes the value $+1$ in a usual Lorentzian regime, and -1 in a Euclidean regime. We show they give essentially different results because of the nature of the singularity at the change surface, and argue that the latter is the better (more physical) choice. It is important to notice that we do not necessarily obtain the correct form of the equations simply by making the obvious transformation $t \rightarrow it$, for this can miss some sign changes resulting from changes in space-time projection factors. Rather we have to carefully examine the field equations and matter equations *ab initio* to determine the correct signs. This is done in a companion paper [8], where a fully covariant formalism is given, based on that in [9], allowing

¹However this doesn't mean we incur with any philosophical implications that may have been drawn from this possibility.

investigation of the effect of a change of signature.

2 Equations

We set up the coordinates and equations in somewhat pedantic fashion; this is essential for clarity later.

2.1 Coordinates and derivatives

As usual the FRW spacetime will be given in comoving coordinates, but now with a lapse function $N(t)$; in terms of coordinates $\{t, r, \theta, \phi\}$ the line element is

$$ds^2 = -N(t)dt^2 + R^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \quad (2)$$

where $N(t)$ is positive for $t > t_0$, negative for $t < t_0$, and zero for $t = t_0$ (the change surface Σ). There is a preferred time parameter σ defined as follows: it is proper time $\tau = \int \sqrt{-ds^2}$ along the fundamental world lines in the Lorentz regime, proper distance $d = \int \sqrt{ds^2}$ along these lines in the Euclidean regime, and continuous across the change surface. The components u^a of the tangent vector of a fundamental world line are defined in terms of σ :

$$u^a = \frac{dx^a}{d\sigma}, \quad x^\nu{}_{,a}u^a = 0 \Rightarrow u^0 = \frac{dt}{d\sigma}, \quad u^\nu = 0 \quad (3)$$

holds everywhere, where u^a is a unit vector almost everywhere:

$$u^a g_{ab} u^b = -\epsilon, \quad (4)$$

with $\epsilon = 1$ in a usual ('Lorentzian') space-time and $\epsilon = -1$ in a 'Euclidean' regime. Using (2) and (3) this relation becomes

$$N(t)\left(\frac{dt}{d\sigma}\right)^2 = \epsilon \Leftrightarrow N(t)(u^0)^2 = \epsilon \quad (5)$$

where

$$\epsilon = \{1 \text{ when } t > t_0, -1 \text{ when } t < t_0, 0 \text{ when } t = t_0\} \quad (6)$$

is required for consistency, clarifying equation (4); in brief, $\epsilon = \text{sign}(N)$ (on the change surface Σ , the vector u^a is non-zero but its magnitude is zero because the metric is degenerate there; everywhere else it is a unit vector). Multiplying (5) by ϵ and using (3) show that in these coordinates

$$\sqrt{|N|}u^a = P(t)\delta_0^a, \quad (7)$$

where $|N| = \epsilon N$ and (from (6))

$$P(t) := \epsilon^2 = |\epsilon| = \{1 \text{ when } t \neq t_0, 0 \text{ when } t = t_0\}. \quad (8)$$

Note that (5) shows $t = \sigma \Rightarrow N(t) = \epsilon \Rightarrow \sqrt{\epsilon N} = P(t)$. Equation (7) shows that

$$t \neq t_0 \Rightarrow u^a = \frac{1}{\sqrt{|N|}}\delta_0^a \quad (9)$$

but does not specify u^a when $t = t_0$ (in this case the equation simply becomes $0 = 0$). However on using continuity in terms of σ , equation (7) shows $t = \sigma \Rightarrow P(t)u^a = P(t)\delta_0^a \Rightarrow u^a = \delta_0^a$, consistently with the definition (3) (which is always valid).

These equations define what we mean by $\dot{f} = f_{;c}u^c = df/d\sigma$, the derivative w.r.t. σ . Letting $f' = df/dt$, the derivative w.r.t. coordinate time t ,

then for every function $f(t)$, by (7)

$$\sqrt{\epsilon N} df/d\sigma = P(t) df/dt \quad (10)$$

(note that both derivatives have the same sign, as t and σ at all times increase in the same direction). This equation is true if $f = \sigma$; if $f = t$; and if $t = \sigma$. It also implies (7), on choosing $f = x^a$ and using (3) It shows that (in agreement with (9) and (1))

$$t \neq t_0 \Rightarrow \frac{df}{d\sigma} = \frac{1}{\sqrt{|N|}} \frac{df}{dt}, \quad (11)$$

but to determine the derivatives on the surface of change we must use the continuity properties of each separately. In particular we can do so when we choose t to be σ : equation (10) then shows we obtain a consistent result: $t = \sigma \Rightarrow P(t)df/dt = P(t)df/d\sigma \Rightarrow df/dt = df/d\sigma$ when f is continuous in terms of σ . From the above it follows that in particular

$$\sqrt{\epsilon N} d\phi/d\sigma = P(t) d\phi/dt, \quad \sqrt{\epsilon N} dR/d\sigma = P(t) dR/dt$$

relate the values of $d\phi/d\sigma$ to $d\phi/dt$ and $dR/d\sigma$ to dR/dt everywhere off the surface of change, but not on it; with the values of these derivatives on the surface of change being determined by continuity properties in terms of the coordinates σ and t respectively.

In these coordinates, the first fundamental form of the surfaces of constant time, including Σ when R and $f_{\mu\nu}$ are continuous there, is

$$h_{ab} = g_{ab} + \epsilon u_a u_b \Rightarrow h_{a0} = 0, \quad h_{\mu\nu} = R^2(t) f_{\mu\nu}(x^\kappa) \quad (12)$$

and their second fundamental form, including Σ when \dot{R} is also continuous there, is

$$K_{ab} = \frac{\dot{R}}{R} h_{ab} \quad (13)$$

(the relations $K_{ab} = u_{a;b} = \frac{1}{3}\Theta h_{ab}$, $\Theta = u^a{}_{;a}$ are valid everywhere off Σ).

2.2 Field equations:

The matter stress tensor will necessarily take the ‘perfect fluid’ form:

$$T^{ab} = (\mu + \epsilon p) u^a u^b + p g^{ab} \quad (14)$$

because of the space-time symmetries (the factor ϵ is required because of (12), see [8]).

There are two independent field equations, which can be obtained directly from the metric (2) on using the standard tensor form of the Einstein equations wherever $N \neq 0$. They can be written as the Raychaudhuri equation

$$3\left(\frac{N'R' - 2NR''}{2NR}\right) = N\frac{\kappa}{2}(\epsilon\mu + 3p) \quad (15)$$

and the Friedmann equation

$$\frac{kN}{R^2} + \frac{R'^2}{R^2} = N\frac{\kappa}{3}\epsilon\mu. \quad (16)$$

The conservation equation is

$$\mu' + (\mu + \epsilon p)\frac{3R'}{R} = 0, \quad (17)$$

and the Friedmann equation is a first integral of the other two equations whenever $R' \neq 0$; thus generically we need only consider the Friedmann and conservation equations. These coordinate forms of the equations can also be obtained directly from their covariant forms (see [8]). We emphasize that in this form they are valid everywhere except on the surface of change itself (on that surface the metric is singular so the inverse metric components, and hence the Christoffel relations, are not well defined).

2.3 Matter Description

We either assume a barotropic perfect fluid:

$$p = p(\mu)$$

or a scalar field ϕ with potential $V(\phi)$. In the latter case

$$\mu = \epsilon \left(\frac{1}{2N} \phi'^2 + V(\phi) \right), \quad p = \frac{1}{2N} \phi'^2 - V(\phi), \quad (18)$$

and the equation of motion is

$$\left(\frac{2N\phi'' - N'\phi'}{2N} \right) + 3 \frac{R'\phi'}{R} + N \frac{\partial V}{\partial \phi} = 0; \quad (19)$$

the conservation equation (17) is a consequence of this equation.

These equations are reasonably well founded in the Lorentzian regime; their meaning in the Euclidean era is open to question. For want of a better way to proceed, we assume the same equations hold there too. This is in accordance with the quantum cosmology approach.

3 Continuous change of signature

A change of signature occurs on a surface $\Sigma : t = t_0$ where $N(t)$ changes from positive to negative. We would like solutions that evolve smoothly through the change, so we look for solutions of (15-17) that can be extended smoothly across the change surface, i.e. R and R' are continuous at Σ .

Assume $N(t)$ is C^1 across the surface of change Σ , where it changes sign; then both $\lim_{t \rightarrow t_0} N(t) = 0$ and $N(t_0) = 0$, and it follows that if $R(t)$ is C^1 then

$$t \rightarrow t_0 \Rightarrow \frac{dR}{dt}(t_0) = \lim_{t \rightarrow t_0} \sqrt{\epsilon N} \frac{dR}{d\sigma} = 0 \quad (20)$$

showing that a maximum, minimum, or point of inflexion of the function $R(t)$ must occur here; the same will clearly be true for any function $f(t)$ that is C^1 and has a regular limit for \dot{f} . However as pointed out before, this is an effect resulting from the degenerate time coordinate. In fact we can have the proper-time derivatives all with a non-zero limit as one approaches the change surface. In particular we can satisfy the limit relation (20) with $d\phi/d\sigma \neq 0$, $dR/d\sigma \neq 0$ in the limit. On the change surface itself, the derivative relation (10) will be identically true. Thus the kinematic requirement is simply that (20) be satisfied, certainly true if $dR/d\sigma$ has a finite, non-zero limit².

² Note that we can use arc length s , defined by $ds = \sqrt{\epsilon N} dt$, as a coordinate across the surface of change, for its definition is singular on that surface; this is why we had to introduce the coordinate σ , which is regular there.

What about the dynamic equations? In the case of a barotropic fluid where μ and R are continuous with R non-zero at Σ , equation (16) requires (20) to hold for consistency. In the case of a scalar field, (16) and (18) show

$$N \rightarrow 0 \Rightarrow \frac{R'^2}{R^2} = \frac{\kappa}{3} \left(\frac{1}{2} \phi'^2 \right) \Rightarrow \phi'^2(t_0) = 0 \quad (21)$$

which will be satisfied due to the equation for ϕ' corresponding to (20). Considering (15), a possible problem arises from the first term on the left, because the surface Σ arises where $N \rightarrow 0$, potentially leading to a divergence in that term. Indeed in terms of the coordinate t the coefficient N'/N will diverge at Σ for many choices of $N(t)$ (indeed probably for all useful choices). However the solutions may still be quite regular across this surface.

The simplest scalar field solutions of the field equations through such a signature change have $\dot{R} = 0 = \dot{\phi}$ at the change, with $\mu + 3\epsilon p \neq 0$ in order to get a turn-around in $R(t)$; and the simplest of these will be exact solutions with

$$\dot{\phi} = 0 \Leftrightarrow \mu + \epsilon p = 0 \Rightarrow \mu = \text{const}$$

everywhere. An interesting example is

$$N(t) = t, \quad k = +1, \quad H \equiv \left(\frac{\kappa}{3} V \right)^{1/2} = \text{const}$$

$$R(t) = \frac{1}{H} \cosh\left(\frac{2}{3} H t^{3/2}\right), \quad \text{for } t \geq 0$$

$$R(t) = \frac{1}{H} \cos\left(\frac{2}{3} H (-t)^{3/2}\right), \quad -\left(\frac{3\pi}{4H}\right)^{2/3} \leq t \leq 0,$$

with Σ at $t = 0$; $R = 1/H$ and $R' = 0$ there for both solutions. The Euclidean regime occurs for $-(3\pi/(4H))^{2/3} \leq t < 0$. This solution, discussed further in section 5, gives the Hartle-Hawking ‘no-boundary’ condition of not having any singularity in the positive definite region.

What is the general criterion for good solutions? From equations (10) and (11), we can re-express the derivatives w.r.t. σ , while still using t as the time parameter in the equations, i.e. we write each function as $f = f(\sigma(t))$. We find for the first term of (15),

$$t \neq t_0 \Rightarrow 3 \frac{N'R'}{2NR} = 3\epsilon \frac{\dot{N}\dot{R}}{2R} = \epsilon \frac{1}{2} \Theta \dot{N}. \quad (22)$$

Thus this term is finite near Σ for those solutions with well-behaved \dot{R} and \dot{N} . It is zero in the limit there if either (1): $\lim_{t \rightarrow t_0} \Theta = 0$, the case of a vanishing second fundamental form for Σ , or (2): $\lim_{t \rightarrow t_0} \dot{N} = 0$; in either case the solution is C^2 across Σ , provided there is no distributional term on Σ , a possibility we examine shortly. This term has a finite jump at Σ if (3): both $\lim_{t \rightarrow t_0} \dot{N}$ and $\lim_{t \rightarrow t_0} \Theta$ are non-zero but finite there. In that case there is a finite discontinuity of R'' there; this is perfectly acceptable at such a jump surface. The solution of the equation (regarded as an equation for R'') will in this case be C^1 but not C^2 there. Essentially the same analysis applies to (19).

As an example of a suitable choice of N , consider

$$N(\sigma) = \sigma - \sigma_0 \quad (23)$$

where Σ occurs at $\sigma = \sigma_0$. Then $\dot{N} = 1$, $\sigma - \sigma_0 = \frac{1}{4}\epsilon(t - t_0)^2$, $N(t) = \frac{1}{4}\epsilon(t - t_0)^2$, $N'(t) = \frac{1}{2}\epsilon(t - t_0)$, and $N''(t) = \frac{1}{2}\epsilon$ (the latter being discontinuous but not occurring in the field equations). Then

$$\frac{R'}{R} = \frac{(t - t_0)}{2} \frac{\dot{R}}{R}, \quad \frac{3N'R'}{2NR} = \epsilon \frac{1}{(t - t_0)} \frac{3R'}{R} = \epsilon \frac{3\dot{R}}{2R}, \quad (24)$$

which is fine as a source term in the second order equation, having at most a finite jump at Σ (if $\dot{R} \neq 0$ there). This corresponds to case (3) above. Secondly, consider

$$N(\sigma) = \epsilon(\epsilon(\sigma - \sigma_0))^{3/2} \quad (25)$$

giving $\dot{N}(\sigma) = \frac{3}{2}(\epsilon(\sigma - \sigma_0))^{1/2}$ which vanishes on Σ . In this case $\sigma - \sigma_0 = 4^{-4}\epsilon(t - t_0)^4$, so $N(t) = 4^{-6}\epsilon(t - t_0)^6$, $N'(t) = 4^{-6}6(t - t_0)^5$, and

$$\frac{3N'R'}{2NR} = \frac{6}{(t - t_0)} \frac{3R'}{R} = \epsilon(\epsilon(\sigma - \sigma_0))^{1/2} \frac{3\dot{R}}{2R}, \quad \frac{R'}{R} = 4^{-3}\epsilon(t - t_0)^3 \frac{\dot{R}}{R} \quad (26)$$

so the potentially troublesome term is continuous, and is zero at Σ even if $\dot{R}|_{\Sigma} \neq 0$. For this choice, $(t - t_0) = 4\epsilon(\epsilon(\sigma - \sigma_0))^{1/4}$. This corresponds to case (2) above. Thirdly, consider

$$N(t) = t - t_0 \quad (27)$$

This is in general not a good choice of lapse function, for then \dot{N} is unbounded at the change surface. However this will cause no problem if $\dot{R} \rightarrow 0$ there; this corresponds to case (1) above, and the specific example just given.

While these conditions apparently will give a nice smooth transition across the jump surface, one must view this with caution: it is possible

for there to be surface layers even though the solution looks quite smooth. Furthermore studying the properties of such solutions is made difficult by the freedom to choose $N(t)$ in an arbitrary way (the nature of the transition is partly masked by our choice of $N(t)$). It is easier to obtain an invariant characterisation of conditions at the jump, and to investigate the question of a distributional contribution to the equations on Σ , if we use the coordinate σ as the ‘time’ coordinate. Before doing so it is necessary to clarify the nature of the singularity at the change surface; we do this next.

4 The change singularity

At the change surface there is a singularity of a subtle kind, somewhat reminiscent of the Taub-NUT singularity where different analytic extensions are possible from the same initial data.

The nature of the problem becomes clear on comparing a single coordinate patch in terms of the coordinate σ across Σ , with a similar patch using a coordinate t with N continuous; the two are in fact incompatible with each other. This can already be seen in the example above: at Σ , $ds/dt \rightarrow 0$ so the transformation is not invertible there and one cannot change from the coordinates (t, r, Θ, ϕ) to coordinates $(\sigma, r, \Theta, \phi)$ there. This is a generic feature of all solutions where there is a change of signature with continuous

$N(t)$, as follows from (11) applied to proper time/distance s :

$$N(t) \text{ continuous} \Rightarrow \frac{ds}{dt} = \sqrt{|N|} \rightarrow 0 \text{ on } \Sigma. \quad (28)$$

The way the coordinates work is as follows:

A] Suppose we can find a coordinate patch U which crosses Σ , with coordinates (t, r, Θ, ϕ) and where the solution is C^1 . We can then find a coordinate patch U_+ that touches but does not include Σ , with coordinates (t_+, r, Θ, ϕ) (in the hyperbolic regime), and a coordinate patch U_- which touches but does not include Σ , with coordinates (t_-, r, Θ, ϕ) (in the Euclidean regime), such that $U = U_+ \cup \Sigma \cup U_-$ (in each domain, we simply have $t_+ = t, t_- = t$).

B] Now we can also define the coordinate patch V_+ to be the same as U_+ but with the coordinates (s_+, r, Θ, ϕ) and a coordinate patch V_- to be the same as U_- but with the coordinates (s_-, r, Θ, ϕ) ; and we can then define the overall patch $V = V_+ \cup \Sigma \cup V_-$ with coordinates $(\sigma, r, \Theta, \phi)$ (in each domain, we simply have $s_+ = \sigma, s_- = \sigma$), where σ is extended across Σ in a continuous manner, as already described. Clearly the point sets covered by U and by V are the same.

C] The issue now is the relation between these descriptions. We can find regular transformations between the two sets of coordinates on $U_+ = V_+$ and on $U_- = V_-$ but not on U or on V (because of (28)); for they represent different differential structures across Σ . In fact there are many families of

differential structures represented by different choices of t in U (depending on the limiting behaviour of $N(t)$ at Σ); each represents a different analytic continuation of the space-time across Σ . The only unique differential structure across Σ is that represented by V , given if we use (as physics suggests) the proper time/distance coordinate s , extended in a continuous manner across Σ to give the coordinate σ . This coordinate is the one we choose here to define continuity in our space-times that change signature, thus defining the manifold structure across Σ . This is then different from the differential structure obtained by *every* choice of a time function t which gives a continuous $N(t)$ across Σ .

It should be noted that this singularity structure is independent of the vanishing or not of the second fundamental form on the join surface. It appears to be inevitable whenever we evolve a Lorentzian to a Euclidean space, or equivalently glue together two such spaces across a join surface Σ with appropriate boundary conditions (which in many ways seems the best description). We develop the latter approach in the next section.

5 Discontinuous change of N

Our aim now is to regularise the equations (15-17), i.e. find a form that is valid in U_+ and U_- but does not run into problems at the change surface. The way to do this is already suggested above: we either

(a) start with a general solution, change to the time coordinate σ , so that then $N(\sigma) = \epsilon(\sigma - \sigma_0)$, by using (10,11) and the consequent equations for second derivatives to change to the variable $\sigma(t)$, and then determine the differential structure across Σ by specifying that the solution be continuous in terms of σ ; or alternatively

(b) make this coordinate choice from the beginning by setting $N = +1$ in the Lorentzian part V_+ and $N = -1$ in the Euclidean part V_- before solving the field equations in each of these regions not intersecting Σ , then joining the solutions across Σ so that they are suitably continuous (by extending the solution in V_- to a solution on the manifold with boundary $\bar{V}_- = V_- \cup \Sigma$ and the solution in V_+ to a solution on the manifold with boundary $\bar{V}_+ = V_+ \cup \Sigma$, in each case setting $N = 0$ on Σ , and then identifying the two boundaries).

In either case, $N \rightarrow \text{const} \neq 0$ as $t \rightarrow t_0$, although $N = 0$ on Σ . Then (10) and (11) do not imply that $d\phi/dt \rightarrow 0$ or $dR/dt \rightarrow 0$ there, for there is no reason then why their limiting values should be zero. Thus setting $\sigma = t$ in these equations is quite consistent and does not lead to the conclusion that (assuming they are continuous) $d\phi/d\sigma = 0 = dR/d\sigma$ at the change surface.

We need, then, to make the choice $N(t) = \epsilon$ in the field equations (15-17), and see what they tell us. Remembering that $\epsilon^2 = 1$ in V_+ and V_- , we find that in these domains, (15) leads to the Friedmann equation

$$\frac{\dot{R}^2}{R^2} = \frac{\kappa}{3}\mu - \epsilon \frac{k}{R^2}, \quad (29)$$

(16) to the Raychaudhuri equation:

$$\frac{\ddot{R}}{R} = -\frac{\kappa}{6}(\mu + 3\epsilon p), \quad (30)$$

and (17) to the conservation equation is

$$\dot{\mu} + (\mu + \epsilon p)\frac{3\dot{R}}{R} = 0. \quad (31)$$

We have a potential problem from the term in N' in equation (15), if we apply this procedure actually on Σ , for the term \dot{N} then becomes $\dot{\epsilon} = 2\delta(t - t_0)$, apparently leading to a distributional (surface layer) term in the equation, which is not physically acceptable because the change surface is a spacelike surface, orthogonal to the fundamental world lines. However it is not sensible to use this process to obtain the equations on Σ , as the original derivation did not hold there. Rather having adopted equations (29-31) in V_+ and V_- , we use the standard analysis of jump conditions [10], based on the work of Darmois [11], to determine whether or not a surface layer is present at the change surface Σ .

In doing so, it is convenient to follow the recent summary by Barrabes [12]. Denote the discontinuity of any function F across Σ by

$$[F] = F^+ - F^-$$

where F^+ (F^-) is the limit of F on approaching a point in Σ from V_+ (V_-). Considering the metric (1) where now $N(t) = \epsilon$, we assume that $R(\sigma)$ and $\dot{R}(\sigma)$ are continuous across Σ , that is,

$$[R] = 0, \quad [\dot{R}] = 0. \quad (32)$$

The spatial coordinates and metric components are unchanged through Σ . Then we find, $[g_{00}] = -[N] = -2$, $[g_{0\alpha}] = 0$, $[g_{\alpha\beta}] = 0$, $[g_{00,a}] = 0$, $[g_{0\alpha,a}] = 0$, $[g_{\alpha\beta,a}] = 0$ (numbering is $\alpha, \beta = 1 - 3$, $a, b, c = 0 - 3$). Consequently the jump in the connection components is zero: $[\Gamma^a_{bc}] = 0$, and so the distributional part of the curvature tensor on Σ vanishes (equations (9)-(12) in [12]). The distributional part of the field equations (equation (16) in [12]) is satisfied if there is no distributional part to the matter stress tensor - the condition we desire, that will be guaranteed by ordinary equations of state for the matter. Assuming this is true, the Einstein field equations in the form (29) and (30) have at most a jump on the surface of change. Thus *using σ as the time coordinate across the surface of change and demanding continuity with respect to this coordinate, conditions (32) ensure there is no distributional part to the field equations.*

There is however a further point we must verify: the conservation law for the stress-energy tensor may still have a distributional part (equation (24) of [12]). We must verify this vanishes. We do so by taking the time derivative of (29) and using (30), taking these equations to be valid across the change surface. We obtain the conservation equation (31), but possibly with a distributional part, which must then be set to zero. This will be done below for each kind of matter considered; this leads to the "jump conditions" we use at such surfaces of change. When we in this way have ensured these surface layer terms are zero, the desired goal has been obtained: a form of the Einstein field equations and conservations equations that can consistently be

continued through the surface where a change of signature takes place.

Finally it must be noted that the proposal above is not the only one that can be made. In effect we have used the Darmois jump condition [11], which is usually equivalent to the Lichnerowicz condition, see Bonnor and Vickers [13]. However at the kind of change surface considered here, *these jump conditions are no longer equivalent* because of the nature of the singularity discussed in the previous section. If one imposes the Lichnerowicz condition - continuity of the full metric across the jump surface, and its first derivatives, necessitating use of the coordinates discussed in section 2 - the results differ from those obtained in this section (they imply a different differential structure across the change surface, as explained above). Instead of obtaining $u^a u_a = \epsilon$ as here, one finds $u_a u^a = 1/\epsilon$, divergent on the change surface, and consequently there is then a distributional contribution to the field equation (S. Hayward, private communication); this leads to the condition $\dot{R} = 0$ on the change surface. Hayward advocates this approach, but we do not agree, both because of its non-uniqueness (many different differentiable structures will be obtained for different choices of $N(t)$) and because of the singular normalisation demanded for u^a in this approach. Thus we claim the Darmois condition used here is the physically relevant one; this gives a unique differential structure, based on proper time and proper distance, and allows $\dot{R} \neq 0$ on the change surface.

5.1 Jump conditions

The question we need to answer then is, given this approach, what are ‘jump conditions’ required at the change surface? To determine them, we consider first a barotropic fluid and then a scalar field.

5.2 Barotropic fluid

The obvious condition is to demand (consistent with (32)) that *across Σ the values of k , μ , R , and dR/ds are continuous*, allowing smooth continuation of the solution from the one side to the other; then p will also be continuous because of the assumed equation of state (which we also take to be the same on both sides). If k/R^2 is non-zero then there is a discontinuity in the sign of $\epsilon k/R^2$ as ϵ changes from +1 to -1. From the Friedmann equation, this implies we must have $k = 0$ (otherwise one of the other quantities would have to be discontinuous there, contrary to our assumption); when this is satisfied the Friedmann equation will go smoothly through the change of signature. Now assuming that the equations (29), (30) are valid everywhere in V , we can check the form of the conservation equation across Σ by taking the derivative of (29) and using (30), remembering that $\dot{\epsilon} = 2\delta$. We find

$$\{[k] = [\mu] = [p] = [R] = [\dot{R}] = 0\} \Rightarrow \dot{\mu} + (\mu + \epsilon p) \frac{3\dot{R}}{R} - 2\delta \frac{k}{R^2} = 0,$$

the last term being the distributional part of the equation; this vanishes if $k = 0$, confirming the validity of the junction condition we have already found.

From the conservation and Raychaudhuri equations, if p is non-zero, then as ϵ changes sign we get discontinuities in $\dot{\mu}$ and d^2R/ds^2 . However if $p = 0$ then both of these quantities are continuous, and indeed in this (Einstein-de Sitter) case the universe goes through the transition with the matter solution ($S(t) = at^{2/3}$) unaffected. This demonstrates that such changes of signature are quite compatible with the field equations, the 3-space and matter variables being as continuous as we could desire. However consequently these solutions do not show any very interesting new behaviour. For that we need to turn to scalar field solutions.

5.3 Scalar fields

We assume a scalar field description applies on both sides of Σ . The relevant equations, obtained from (18), (19), are now

$$\mu = \frac{1}{2}\dot{\phi}^2 + \epsilon V(\phi), \quad p = \epsilon \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad (33)$$

$$\ddot{\phi} + 3\frac{\dot{R}\dot{\phi}}{R} = -\epsilon\frac{\partial V}{\partial\phi}, \quad (34)$$

the latter being the Klein-Gordon equation. The Friedmann equation becomes

$$\frac{\dot{R}^2}{R^2} = \frac{\kappa}{6}\dot{\phi}^2 + \epsilon\left(\frac{\kappa}{3}V(\phi) - \frac{k}{R^2}\right) \quad (35)$$

and the Raychaudhuri equation becomes

$$\frac{\ddot{R}}{R} = -\frac{\kappa}{3}(\dot{\phi}^2 - \epsilon V(\phi)). \quad (36)$$

The natural requirements in this case (compatible with (32)) are that R , \dot{R} , ϕ , $\dot{\phi}$, and $V(\phi)$ are continuous across Σ (the point about $V(\phi)$ being that we could have different functional forms for the potential on the two sides, so that even if ϕ is the same on both sides of Σ we must check that $V(\phi)$ is also). From the Friedmann equation, this requires that on Σ ,

$$\kappa V(\phi) = \frac{3k}{R^2} \quad (\Rightarrow k > 0) \quad (37)$$

$$\frac{\dot{R}^2}{R^2} = \frac{\kappa}{6} \dot{\phi}^2, \quad (38)$$

Assuming the Friedmann equation holds true as one approaches the transition, the first equation can be interpreted as *the criterion deciding when the transition takes place*. The second equation will then be *automatically satisfied at that time* (because the Friedmann equation was satisfied near there). From the Raychaudhuri equation (36) we see that \ddot{R}/R will suffer a discontinuity through the change of sign of ϵ : $\dot{\phi}$ and V being continuous,

$$\left[\frac{\ddot{R}}{R} \right] = \frac{2\kappa}{3} V(\phi) \quad (39)$$

As in the case of the barotropic fluid, we can assume the equations (34)-(36) hold across the surface of change, and check for the occurrence of a surface layer term in the conservation equation by taking the derivative of (35) and using (34), (36). We find

$$\{[k] = [\phi] = [p\dot{h}i] = [R] = [\dot{R}] = 0\} \Rightarrow \dot{\mu} + (\mu + \epsilon p) \frac{3\dot{R}}{R} - \delta \left(\kappa V(\phi) - \frac{3k}{R^2} \right) = 0,$$

so such a term does not occur if and only if (37) is satisfied, confirming that this is indeed the correct change condition to choose.

In non-trivial cases at least one of μ and p , while remaining finite, will have a jump in value across the surface Σ ; however the kinetic and potential energies of the field are both continuous there. Perhaps the jump in μ and p is not so serious as one might at first think in view of the fact that life for the observer who would measure such effects becomes somewhat transformed as she changes from a Lorentz to a Euclidean metric; indeed the whole concept of matter changes at this epoch, and observers cannot exist at earlier times. However the 3-space itself undergoes the transition without any disaster (the first and second fundamental forms being continuous), so in the ADM and quantum cosmology spirit, where the 3-metric and the matter variables are regarded as the relevant dynamical variables, the solution can be regarded as quite regular. As explained above, the conditions chosen are the generalisation to the present situation of the standard Darmois conditions for non-existence of a surface layer. They are supported by the fact that the equations for those geodesics that are timelike in the Lorentz domain can then be continued without change of form across the surface of change and with coefficients that are continuous there, so their solutions will go smoothly across also, in the sense of each geodesic $x^a(v)$ being a C^2 (but generally not C^3) curve. Thus the affine structure is regular across the change surface (as follows from the fact that there is no jump in the affine connection components in the chosen coordinates).

The implication of the above is that one can take as the criterion for

where there should be a Lorentzian regime

$$\kappa V(\phi) > \frac{3k}{R^2} \Rightarrow \epsilon = 1, \quad (40)$$

and the criterion for where there should be a Euclidean regime

$$\kappa V(\phi) < \frac{3k}{R^2} \Rightarrow \epsilon = -1, \quad (41)$$

both requirements leading to (and required for) the condition

$$\frac{\dot{R}^2}{R^2} > \frac{\kappa}{6} \dot{\phi}^2, \quad (42)$$

ensuring that \dot{R}^2 is positive. Indeed one can regard this as the rationale for the change: where the form of $V(\phi)$ is such that at some epoch the classical field equations would normally lead to the conclusion that \dot{R}^2 will become imaginary, instead of assuming quantum tunnelling is taking place (as in the quantum cosmology interpretation) we deduce that a change of signature is required in the classical solution. In the case when the no-rolling condition $\dot{\phi} = 0$ is satisfied, (40) and (41) are precisely the criteria required for \dot{R}^2 to always be positive; when rolling takes place, the criteria should be (40) and (41), to ensure both that the required change of signature takes place smoothly (the change-over takes place when (37), (38) are satisfied) and that \dot{R}^2 is positive.

Finally we note that at least formally an analytic continuation is possible across the surface, by change of time to complex time: $t \rightarrow it$. Specifically, considering the Klein-Gordon equation (34), the Friedmann equation (35),

and the Raychaudhuri equation (36), if we have a solution $R(t)$, $\phi(t)$ of these equations for given ϵ , k and $V(\phi)$, then $\tilde{R}(t) = R(it)$, $\tilde{\phi}(t) = \phi(it)$ will also be a solution of the equations for $\tilde{\epsilon} = -\epsilon$ and with the same k , $V(\phi)$; similar methods have been used in the quantum cosmology context, see e.g. [14]. However the real solution that continues across Σ with the desired continuity properties may not be that obtained directly by this method. In the present context we can use an alternative method of analytic continuation: namely introducing a power-series description for $R(t)$ in each domain and matching the series across the jump surface, with the minimum change necessary to accommodate the change of signature. The way this can be done will be demonstrated below.

5.4 Simplest cases

The simplest examples are the solutions with $\dot{\phi} = 0 \Leftrightarrow p = -\mu$, that is the ‘cosmological constant’ (no rolling) case, which are allowed as special solutions of the equations (for particular $V(\phi)$). Then p is continuous and the jump conditions show $\dot{R} = 0$ on Σ , while if $V(\phi) \neq 0 (\Rightarrow k \neq 0)$ then \ddot{R} is non-zero and changes sign there; that is, *the radius function has a point of inflection at the change-over.*

Exact solutions of this form can be obtained by solving the field equations in each regime, ensuring the jump conditions are satisfied. A particularly interesting one is that discussed in the third section, but now obtained in

simpler coordinates: we have

$$k = 1, \quad \phi = \text{const}, \quad H = \left(\frac{\kappa V}{3}\right)^{1/2} = \text{const},$$

$$\epsilon = -1, \quad R(t) = \frac{1}{H} \cos Ht, \quad \text{for } -\pi/(2H) \leq t \leq 0;$$

$$\epsilon = 1, \quad R(t) = \frac{1}{H} \cosh Ht \quad \text{for } t \geq 0,$$

with Σ at $t = 0$ and the Euclidean regime occurring for $-\pi/(2H) \leq t < 0$. This is exactly what is desired from the Hartle-Hawking viewpoint, corresponding to their ‘no-boundary’ prescription for the initial phase of the universe; an inflationary phase emerges from a Euclidean regime which is exactly a 4-sphere, and so is without boundary [see Figure 1]. The universe in the Euclidean phase ‘is’ (there is a 4-space there), but does not *exist* (for there is no time there), and for example one cannot perform experiments there (no observer can measure or act there). Time begins at the surface of transition Σ (given in these coordinates by $t = 0$) but the universe does not begin there, for that surface is quite regular, and the universe extends through it, indeed *the universe has no beginning* because the positive definite pre-expansion state is perfectly regular: there is no singularity or boundary to space-time there.

We can tie this in to the usual definition of singularities by observing that these spaces are *geodesically complete*. When a timelike geodesic $x^a(v)$ crosses the surface Σ from the Lorentzian region, with v an affine parameter, the curve $x^a(v)$ and its tangent vector $X^a = dx^a/dv$ are continuous there, although there is a discontinuity in its magnitude $E \equiv g_{ab}X^aX^b$ (due to the

change in the metric signature). Nevertheless the geodesic can be continued through without problem, the geodesic equation (expressed as the equation of parallel transfer) being continuous through the jump, and giving unique answers to continuation from one side to the other; the affine structure is perfectly regular. Geodesic completeness follows because the Euclidean regime is just part of the standard 4-sphere S^4 .

Consider for example the integral curves of the 4-velocity u^a : these are geodesics which (going into the past) make a transition from the Lorentz to the Euclidean regime by passing through a point q on Σ . They then move through this S^4 , passing through the point r where $t = -\pi/(2H)$ (a standard spherical coordinate singularity at the South Pole), intersecting the surface Σ again at the point q' on that 3-sphere antipodal to the point q of entry, and then proceeding up in the Lorentzian regime in the future time direction [see Figure 1]. Similarly all timelike geodesics in the Lorentzian regime will enter the Euclidean regime through Σ and then re-emerge into the Lorentzian regime. Thus these space-times are singularity-free.

It is interesting to point out here that in the Lorentzian regime the energy conditions are violated (as in all inflationary solutions), but they are obeyed in the Euclidean regime. Thus the avoidance of a density singularity there is not because of energy violation, but rather because the Hawking-Penrose singularity theorems do not apply in a Euclidean regime (they depend on various causal properties that do not hold there). Furthermore because there are

no horizons in the early (Euclidean) phase, there may be no horizon problem in this universe: in the early domain there is a coordinate that is destined to become time, and a variation (to which we can if we wish still append the name ‘evolution’) of variables in terms of that coordinate, which ensures that their values at different positions on the change surface are not independent of each other. This also means that it is possible the monopole problem (and the prediction of other topological defects or textures) is side-stepped.

In this simplest model, the universe cannot emerge from the inflationary phase in the hyperbolic regime because the scalar field is not rolling. Generalisations discussed below can fix up this problem.

5.5 More general solutions

More generally we can use the Ellis-Madsen procedure [15] for generating exact solutions of the equations when $V(\phi)$ is unknown, by using the equations

$$\kappa V = \epsilon \left(\frac{\ddot{R}}{R} + 2 \frac{\dot{R}^2}{R^2} \right) + \frac{2k}{R^2} \quad (43)$$

$$\kappa \dot{\phi}^2 = 2 \left(\frac{\dot{R}^2}{R^2} - \frac{\ddot{R}}{R} \right) + \epsilon \frac{2k}{R^2} \quad (44)$$

that are algebraically equivalent to (35), (36). Choose the desired behaviour of the solution $R(t)$; the second equation determines $\dot{\phi}(t)$, and so $\phi(t)$ and hence $t(\phi)$; the first then gives $V(t)$ and so determines $V(\phi)$ from $V(t) = V(t(\phi))$. Then the Klein-Gordon equation and all the Einstein equations will be satisfied [15]. Clearly we can apply this procedure in each regime, and

join the solutions across a suitable boundary. Substituting into (37) from (43) or into (38) from (44), we find that the jump condition is satisfied when

$$\frac{2\dot{R}^2}{R^2} + \frac{\ddot{R}}{R} = \epsilon \frac{k}{R^2} \quad (45)$$

Thus this is an alternative form of criterion for the existence of a change surface Σ .

We would like to find solutions like the ‘no-boundary’ one discussed above, but with $\dot{\phi}(t) \neq 0$, to allow an exit from inflation. To obtain the 4-sphere feature locally at the point r where $R \rightarrow 0$ requires that μ goes towards a constant as there (making the metric locally like a 4-sphere: the singularity at $R \rightarrow 0$ is only an apparent one). Thus we need $\mu + \epsilon p = 0 \Leftrightarrow \dot{\phi} = 0$ at r . This is possible with $\dot{\phi} \neq 0$ away from r in a neighbourhood of that point, provided $\partial V/\partial \phi \neq 0$ at r .

To construct a specific example, it is convenient to introduce a new ‘time’ coordinate y such that the coordinate singularity where $R = 0$ occurs at $y = 0$. Thus we choose $y \equiv t + \alpha$, so $y = 0 \Leftrightarrow R = 0 \Leftrightarrow t = -\alpha$, and the change-over surface Σ is $t = 0 \Leftrightarrow y = \alpha$ (in the previous example, $\alpha = \pi/(2H)$).

We consider the case where the scale function in the Euclidean domain is

given by the first two terms in the power series for $(1/H)\sin Hy$, that is,

$$R(y) = y(1 - \frac{1}{6}H^2y^2), \quad \epsilon = -1, \quad \text{for } 0 \leq y \leq \alpha \Leftrightarrow -\alpha \leq t \leq 0. \quad (46)$$

To find V , we have to solve (43), (44) in the Euclidean regime with 'time' coordinate y . They show that

$$\kappa \dot{\phi}^2 = \frac{1}{6} \frac{H^4 y^2}{(1 - \frac{1}{6}H^2 y^2)^2}, \quad \kappa V = 3H^2 \frac{(1 - \frac{2}{9}H^2 y^2)}{(1 - \frac{1}{6}H^2 y^2)^2},$$

giving the correct 4-sphere ('no-boundary') limiting behaviour as $y \rightarrow 0$. We can integrate and invert to find $V(\phi)$, as explained above, but we do not need to use the resulting forms to obtain much of what we want to know. Putting (46) into (45), we can solve for the value of y such that change condition (45) holds; this value is α . We find

$$H^2 \alpha^2 = 3/2 \quad \Leftrightarrow \quad \alpha = \frac{1}{H} \sqrt{\frac{3}{2}}.$$

This lets us determine the values of R , \dot{R} , \ddot{R} , $V(\phi)$, $\dot{\phi}$, and any desired higher time derivatives at the jump surface (insert $y = \alpha$ in the formulae above, or their time derivatives).

This gives us the information we need to match the solutions in the two domains. We represent the solution in the hyperbolic domain by a power series:

$$R(t) = \sum_{n=0}^{\infty} a_n t^n, \quad \epsilon = +1, \quad \text{for } t \geq 0, \quad (47)$$

We now determine the constants a_0 and a_+ so that the values of $R|_0$, $\dot{R}|_0$, are continuous at $t = 0$, and a_- so that \ddot{R} obeys condition (39); we find

$$a_0 = \frac{3}{4}\alpha, \quad a_1 = \frac{1}{4}, \quad a_2 = \frac{7}{18}H^2\alpha$$

Given these values, then $\dot{\phi}$ and $V(\phi)$ will also be continuous, and the jump conditions (37) and (38) will hold as limits from both the right and the left.

What then about all the higher order constants in (47)? They are not constrained by the jump condition (37), or equivalently (45); their choice depends on what kind of evolution we wish to specify. One possibility is that we choose any evolution we desire for $t > 0$, then using the Ellis-Madsen approach [15] to determine the potential $V(\phi)$ that is effective for motion in the hyperbolic regime $t > 0$. The alternative is that we require the potential to have the same form on both sides (corresponding to the idea of analytic continuation), and determine the coefficients from this condition. How do we do so? We take time derivatives of the equations already written down, which enable us to determine the change in all the higher order derivatives in terms of known lower order derivatives and other functions that are continuous. For example, taking a time derivative of (36) shows that

$$\frac{1}{R} \frac{d^3 R}{dt^3} = \frac{5}{3} \frac{\dot{R}}{R} \kappa \dot{\phi}^2 + \epsilon \left(\frac{\kappa V}{3} \frac{\dot{R}}{R} + \kappa \dot{\phi} \frac{\partial V}{\partial \phi} \right) \quad (48)$$

Just as (36) lead to the condition (39), this will lead to the condition

$$\left[\frac{1}{R} \frac{d^3 R}{dt^3} \right] = 2 \left(\frac{\kappa V}{3} \frac{\dot{R}}{R} + \kappa \dot{\phi} \frac{\partial V}{\partial \phi} \right) = 2 \epsilon \left(\frac{1}{R} \frac{d^3 R}{dt^3} - \frac{5 \dot{R}}{3 R} \kappa \dot{\phi}^2 \right) \quad (49)$$

for the jump in the third derivative, which we can use to determine a_3 from the known value of the third derivative in the Euclidean domain, provided we know that $\partial V / \partial \phi$ takes the same value on both sides, which we assume because of our analyticity requirement. Note that we do not actually have to

know the value of $\partial V/\partial\phi$ to determine the jump, because by using (48) it can be re-expressed in terms of the third time derivative. Similarly taking higher derivatives will successively determine each of the higher coefficients a_n from the assumption that the higher derivatives of $V(\phi)$, evaluated at the jump surface, are the same on both sides. In practice the computations involved rapidly get messy, but the principle is quite straightforward and leads to a unique power series extension across the boundary.

The interesting point then is that this procedure does not always lead to what one might have at first expected. Specifically, in the example given above, the fourth and higher terms in (47) do not vanish when we follow this procedure (the fourth term is non-zero; we have not carried out a detailed check of terms beyond the fourth, but it seems highly unlikely they will vanish). Thus the procedure of continuation used here, developing a power series that corresponds to assuming the same functional form $V(\phi)$ holds on both sides of the surface Σ and continuing the solution smoothly through, cannot correspond to a simple complex time substitution in (46), for that would lead to all the higher order terms vanishing in (47).

In any case, whatever continuation method we use, this space-time has the same desired ‘no-boundary’ properties for $0 < t$ as in the previous case (for in the Euclidean regime, $R(y)$ is just the first couple of terms of the power series for $\sin y$, giving the desired 4-sphere behaviour at $y = 0$); and in the Lorentzian regime the solution ends up in a power-law inflation. We

can obtain exact scalar field solutions of this kind for suitable choices of the potential $V(\phi)$, and they will have $\dot{\phi} \neq 0$ almost everywhere, so there will be no problem about the end of inflation (for values of ϕ corresponding to some late time, we assume the potential has a sharp drop like that of the potential in ‘new inflation’, leading to the conversion of this field into radiation).

This is just one of many choices one could have made; we could for example have included any desired number of terms from the power series for $\sin y$ in (46) and then have proceeded as above, so obtaining an infinite family of different such solutions (each with a different $V(\phi)$). Clearly there is no uniqueness imposed by the ‘no-boundary’ condition by itself; to get uniqueness one must impose extra restrictions (e.g. additionally assuming maximisation of some quantity associated with the solutions for a fixed $V(\phi)$). We can also find solutions with cusps or other singularities in the positive definite region; however they do not correspond to the Hartle-Hawking ‘no-boundary’ proposal.

6 Conclusion

The prime point arising in this paper is that

the Einstein field equations by themselves do not determine the space-time signature; that is imposed as an extra assumption.

The consequence is that if we formulate the equations in a (3+1) way,

we can construct exact solutions of the field equations where there is a change of signature on a spatial hypersurface Σ , with the 3-space metric and its first derivatives behaving regularly there (the solution is without surface layer terms).

This enables us to construct exact classical solutions that are analogues of the Hartle-Hawking proposal, where space-time has a Euclidean signature at very early times, and in this way avoids a singular origin to the universe (in effect, time has an origin some distance from the initial events in the universe, which constitute a Euclidean regime where time does not exist, and without boundary). The space-time is singularity-free in the sense that all geodesics are complete.

We can regard the models either (1) as a representation of what is predicted from the Hartle-Hawking studies of the wave function of the universe, providing classical models of the quantum gravity regime; or (2) we could take them as indicating an alternative possibility: if such a change of signature took place in the early universe this side of the Planck epoch, for example at the GUTS energy, we avoid the need for a quantum gravity domain (the density would never reach the Planck value) while simultaneously avoiding the implications of the Hawking-Penrose singularity theorems (for these do not apply in a Euclidean regime). In this case the Euclidean domain allows us to avoid the introduction of complex time and quantum tunnelling, while maintaining a positive kinetic energy term \dot{R}^2 . A criterion has been found (in the case of scalar field solutions) for when there should be a Euclidean

domain and when the change of signature should occur, effectively comparing the potential energy of the scalar field with the spatial curvature. Thus the epoch of change-over is determined by the scalar field potential. Whether it would take place at the Planck or GUTs era (or some other time) would therefore depend on the nature of the scalar fields that are dynamically important. What happens to the usual uniqueness theorems for solutions of the Einstein equations, in these models? They still apply, in a slightly extended sense: there is a unique evolution from given initial data for given $V(\phi)$, if we are given a prescription as to where signature changes take place; and such a prescription has been given.

This idea presented here is clearly related to those recent proposals where the metric is allowed to be degenerate in a quantum gravity epoch (see Horowitz [16] and papers quoted there) and papers pointing out that interesting effects such as particle production [17] and a change of topology [16,18] can occur if a signature change takes place. No specific mechanism has been suggested here that would implement such a classical change of signature, and this can be regarded as a weakness of the present proposal. However in the usual classical analyses of Robertson-Walker universes, no mechanism is specified that will maintain the signature as it was initially; and this is equally a weakness of those analyses.

Whatever interpretation we might adopt, these models are not unique; there are various ones that are regular, as well as ones where there is a sin-

gularity in the Euclidean region. The interesting point is that we are able to find such classical solutions.

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FIGURE 1

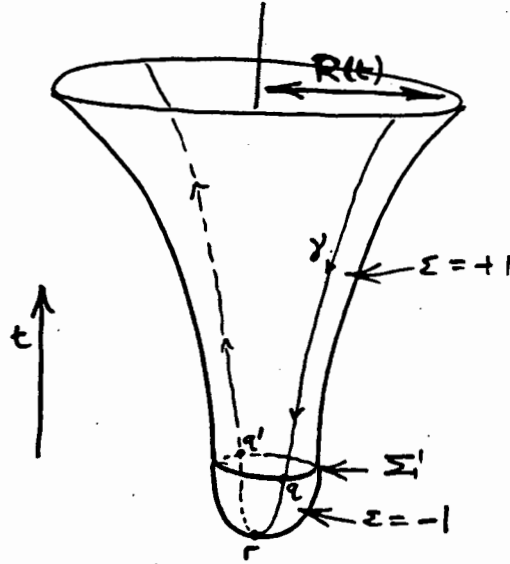


Figure 1: Imbedding diagram showing change of radius $R(t)$ with coordinate time t for a classical universe obeying the 'no-boundary' condition. The space-sections are closed because $k = +1$. The top ($\epsilon = +1$) is an inflationary expansion of a Lorentzian space-time region; the bottom ($\epsilon = -1$) is a Euclidean region, which is like a 4-sphere near the South Pole r . These regions are joined to each other smoothly at the surface Σ where time begins. The universe is geodesically complete, for example the geodesic γ shown starts in the hyperbolic regime and passes through q on Σ , then through the coordinate singularity r at the South Pole in the Euclidean regime, and finally re-emerges in the hyperbolic regime through q' on Σ .